Math 416 solutions to Problem Set 3:

1. you can either apply the definition of derivative to find f' and f'' or use the Cauchy-Riemann equations.

(a) $f(z) = iz^2+1 = i(x+iy)^2+1 = \underbrace{(1-2xy)}_{u(x,y)} + i\underbrace{(x^2-y^2)}_{v(x,y)}$

$$\left.\begin{aligned} u_x &= -2y = v_y \\ u_y &= -2x = -v_x \end{aligned}\right\}$$ and u_x, u_y, v_x, v_y are continuous so f' exists and

$$f' = u_x + iv_x = -2y + i(+2x) = 2iz$$

And $2iz = \underbrace{-2y}_{u_1} + i\underbrace{(2x)}_{v_1}$

$$\left.\begin{aligned} \frac{\partial}{\partial x}u_1 &= 0 = \frac{\partial}{\partial y}v_1 \\ \frac{\partial}{\partial y}u_1 &= -2 = -\frac{\partial}{\partial x}v_1 \end{aligned}\right\}$$ and all these partial derivaties are continuous, so

$$f'' = \frac{\partial}{\partial x}u_1 + i\frac{\partial}{\partial x}v_1 = 2i$$

(b) $f(z) = e^{-x}e^{-iy} = e^{-x}(\cos y - i\sin y) = \underbrace{e^{-x}\cos y}_{u(x,y)} + i\underbrace{(-e^{-x}\sin y)}_{v(x,y)}$

$$\left.\begin{aligned} u_x &= -e^{-x}\cos y = v_y \\ u_y &= -e^{-x}\sin y = -v_x \end{aligned}\right\}$$ all the partial derivaties are continous so $f' = u_x + iv_x$

$$= \underbrace{-e^{-x}\cos y}_{u_1} + i\underbrace{e^{-x}\sin y}_{v_1} = -f$$

Similarly

$$\left.\begin{aligned} \frac{\partial}{\partial x}u_1 &= e^{-x}\cos y = \frac{\partial}{\partial y}v_1 \\ \frac{\partial}{\partial y}u_1 &= e^{-x}\sin y = -\frac{\partial}{\partial x}v_1 \end{aligned}\right\}$$ so $f'' = e^{-x}\cos y - ie^{-x}\sin y = f$

(you could also say since $f' = -f$

$f'' = (-f)' = -f' = f$)

2. (a) We have

$$\frac{\partial g}{\partial \bar{z}} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{\partial g}{\partial x} \times \frac{1}{2} + \frac{\partial g}{\partial y} \times \frac{-1}{2i}$$

$$= \frac{1}{2}\left(\frac{\partial g}{\partial x} + i\frac{\partial g}{\partial y}\right)$$

(b) If $f(z) = u(x,y) + iv(x,y)$,

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = \frac{1}{2}\frac{\partial f}{\partial x} + \frac{i}{2}\frac{\partial f}{\partial y}$$

$$= \frac{1}{2}(u_x + iv_x) + \frac{i}{2}(u_y + iv_y)$$

$$= \frac{1}{2}\left[(u_x - v_y) + i(v_x + u_y)\right].$$

If the Cauchy-Riemann equation $u_x = v_y$, $u_y = -v_x$ are satisfied, this tells us that $\frac{\partial f}{\partial \bar{z}} = 0$

(3) (a) $f(z) = \begin{cases} \frac{\bar{z}^2}{z} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$

Now $f(z) = \underbrace{\frac{x^3 - 3xy^2}{x^2+y^2}}_{u} + i\underbrace{\frac{y^3 - 3x^2y}{x^2+y^2}}_{v}$

when $z \neq 0$. We have

$$u_x(0,0) = \lim_{\Delta x \to 0} \frac{u(0+\Delta x, 0) - u(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

$$u_y(0,0) = \lim_{\Delta y \to 0} \frac{u(0, 0+\Delta y) - u(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{0}{\Delta y} = 0$$

$$v_x(0,0) = \lim_{\Delta x \to 0} \frac{v(0+\Delta x, 0) - v(0,0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

$$v_y(0,0) = \lim_{\Delta y \to 0} \frac{v(0, 0+\Delta y) - v(0,0)}{\Delta y} = \lim_{\Delta y \to 0} \frac{\Delta y}{\Delta y} = 1,$$

So at $(0,0)$ $v_x = -u_y$ $v_y = u_x$

If $(a,b) \neq (0,0)$

$$u_x(a,b) = \frac{(3a^2 - 3b^2)(a^2+b^2) - (a^3 - 3ab^2)(2a)}{(a^2+b^2)^2}$$

So u_x exist, and similarly v_x, u_y, v_y exist.

Note that although the partial derivatives exist at every point, and the Cauchy-Riemann equations are satisfied, we cannot conclude f is differentiable at the origin because u_x, u_y, v_x, v_y are not continuous. (you can check for example that $\lim_{(x,y)\to(0,0)} u_x(x,y)$ does not exist.)

(b) $$\lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x \to 0} \frac{x - 0}{x} = 1$$

But
if $x=y$

$$\lim_{x\to 0} \frac{f(x,x) - f(0,0)}{x+ix} = \lim_{x\to 0} \frac{-x-ix}{x+ix} = -1$$

(4). $f(z) = \underbrace{3x+y}_{u} + i\underbrace{(3y-x)}_{v}$

f is defined everywhere and

$$\left.\begin{array}{l} u_x = 3 = v_y \\ u_y = 1 = -v_x \end{array}\right\} \Rightarrow f \text{ analytic everywhere.}$$

$f(z) = \dfrac{3z+1}{z(z^2+4)}$ f is not defined at $z=0, 2i, -2i$

Everywhere else the numerators and the denominator ary polynomial functions and so analytic.
Thus f is analytic other that $z=0, 2i, -2i$

$f(z) = \underbrace{(y^2-x^2)}_{u} + i\underbrace{(2xy)}_{v}$

$u_x = -2x$ $v_y = 2x$ $u_x = v_y$ only when $x=0$
$u_y = 2y$ $v_x = 2y$ $u_y = -v_x$ only when $y=0$

So the derivative of f exist only at the origin,
so f is Not analytic at any point (we defined

f to be analytic at a point if the derivative exist in a disc around the point).

5. (a) Using Polar coordinates:

$$g(z) = \underbrace{\ln r}_{u} + i\underbrace{\theta}_{V} \qquad 0 < \theta < 2\pi, \quad 0 < r$$

$$u_r = \frac{1}{r} \qquad v_\theta = 1$$
$$u_\theta = 0 \qquad v_r = 0$$

$\Rightarrow$ the Cauchy-Riemann equations are satisfied

Hence g is analytic with derivative

$$g' = e^{-i\theta}(u_r + i v_r) = e^{-i\theta}\left(\frac{1}{r} + i \times 0\right) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

(b): $(x+iy)^2 + 1 = (x^2 - y^2 + 1) + 2ixy$

If $2xy = 0$, then $x=0$ or $y=0$. So if $x>0$ $y>0$,

z^2+1 is not on the real line, so we can choose its argument to be between 0 and 2π.

So we have the composition of 2 analytic functions which is analytic, and the chain rule says:

$$G'(z) = g'(z^2+1) \times 2z = \frac{2z}{z^2+1}$$

6. $u(x,y) = 2x - x^3 + 3xy^2$.

$$\left.\begin{aligned} u_x &= 2 - 3x^2 + 3y^2 \qquad & u_{xx} &= -6x \\ u_y &= 6xy & u_{yy} &= 6x \end{aligned}\right\} \text{ so } \quad u_{xx} + u_{yy} = 0$$

To find a harmonic conjugate $v(x,y)$, we start with $u_x(x,y) = 2 - 3x^2 + 3y^2$. Now

$$u_x = v_y \Rightarrow v_y = 2 - 3x^2 + 3y^2 \Rightarrow v(x,y) = 2y - 3x^2y + y^3 + \phi(x)$$

Then

$$u_y = -v_x \Rightarrow 6xy = 6xy - \phi'(x) \Rightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c.$$

consequently,

$$v(x,y) = 2y - 3x^2y + y^3 + c$$