

Math 416 Complex variables

Solutions to Problem Set 5

1. (i) We have $(\frac{1}{t} - i)^2 = (\frac{1}{t^2} - 1) - i(\frac{2}{t})$. Therefore,

$$\begin{aligned}\int_1^2 (\frac{1}{t} - i)^2 dt &= \int_1^2 (\frac{1}{t^2} - 1) dt - i \int_1^2 \frac{2}{t} dt \\ &= (\frac{-1}{t} - t)|_1^2 - i(2 \log t)|_1^2 \\ &= \frac{-1}{2} + 1 - 2 + 1 - i(2 \log 2 - 2 \log 1) \\ &= \frac{-1}{2} - 2i \log 2.\end{aligned}$$

- (ii) Assume $z = a + ib$. Since $\frac{d}{dt} e^{zt} = ze^t$, we have $\frac{d}{dt} \frac{-e^{-zt}}{z} = e^{-zt}$, and so

$$\begin{aligned}\int_0^\infty e^{-zt} dt &= \lim_{s \rightarrow \infty} \int_0^s e^{-zt} dt \\ &= \lim_{s \rightarrow \infty} \frac{-e^{-zt}}{z} \Big|_0^s \\ &= \lim_{s \rightarrow \infty} (\frac{1}{z} - \frac{e^{-sz}}{z}) \\ &= \lim_{s \rightarrow \infty} (\frac{1}{z} - \frac{e^{-sa}(\cos(bs) - i \sin(bs))}{z}) \\ &= \frac{1}{z}.\end{aligned}$$

2. If $m = n$, then we have

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \int_0^{2\pi} 1 d\theta = \theta \Big|_0^{2\pi} = 2\pi.$$

If $m \neq n$, then we have

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \int_0^{2\pi} e^{i(m-n)\theta} d\theta = \frac{e^{i(m-n)\theta}}{i(m-n)} \Big|_0^{2\pi} = \frac{e^{i(m-n)2\pi} - 1}{i(m-n)} = 0.$$

3. Write $f(z) = u(x, y) + iv(x, y)$ and $z(t) = x(t) + iy(t)$. Then we have,

$$w(t) = \overline{u(x(t), y(t))} + iv(x(t), y(t)).$$

So by the chain rule

$$\begin{aligned} \frac{d}{dt} w(t) &= \left(\frac{d}{dt} u(x(t), y(t)) \right) + i \left(\frac{d}{dt} v(x(t), y(t)) \right) \\ &= (u_x x' + u_y y') + i(v_x x' + v_y y') \\ &= (u_x x' - v_x y') + i(v_x x' + u_x y') \quad (\text{by Cauchy-Riemann equations}) \\ &= (u_x + iv_x)(x' + iy') \\ &= (u_x + iv_x) z'(t). \end{aligned}$$

On the other hand, if $f = u + iv$ is analytic, then $\frac{d}{dz} f = u_x + iv_x$, so

$$\frac{d}{dt} w(t) = f'(z(t)) z'(t).$$

4. (i) Let C_1 be the line segment from 0 to 1, C_2 be the line segment from 1 to $i + 1$, C_3 the the line segment from $i + 1$ to i , and C_4 be the line segment from i to 0. Then C_1 is parametrized by $z(t) = t, 0 \leq t \leq 1, z'(t) = 1$, and

$$\int_{C_1} f(z) dz = \int_0^1 \pi e^{\pi t} dt = e^{\pi t} \Big|_0^1 = e^\pi - 1.$$

C_2 is parametrized by $z(t) = 1 + it, 0 \leq t \leq 1, z'(t) = i$, and

$$\int_{C_2} f(z) dz = \int_0^1 \pi i e^{\pi(1-it)} dt = -e^{\pi(1-it)} \Big|_0^1 = e^\pi - e^{\pi(1-i)} = 2e^\pi.$$

C_3 is parametrized by $z(t) = (1-t) + i, 0 \leq t \leq 1, z'(t) = -1$, and

$$\int_{C_3} f(z) dz = \int_0^1 -\pi e^{\pi(1-t-i)} dt = e^{\pi(1-t-i)} \Big|_0^1 = e^{-\pi i} - e^{\pi(1-i)} = e^\pi - 1.$$

C_4 is parametrized by $z(t) = (1-t)i, 0 \leq t \leq 1, z'(t) = -i$, and

$$\int_{C_4} f(z) dz = \int_0^1 -i\pi e^{\pi(-1+t)i} dt = -e^{\pi(-1+t)i} \Big|_0^1 = e^{-\pi i} - 1 = -2.$$

Thus

$$\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz + \int_{C_3} f(z) dz + \int_{C_4} f(z) dz = 4e^\pi - 4.$$

- (ii) Let C_1 be the arc from $-1-i$ to 0 along $y = x^3$ and C_2 the arc from 0 to $1+i$ along $y = x^3$. Then C_1 is parametrized by $z_1(t) = t + it^3, -1 \leq t \leq 0$, and C_2 is parametrized by $z_2(t) = t + it^3, 0 \leq t \leq 1$. We have $z_1'(t) = z_2'(t) = 1 + 3it^2$.

$$\int_{C_1} f(z) dz = \int_{C_1} 1 dz = \int_{-1}^0 (1 + 3it^2) dt = (t + it^3) \Big|_{-1}^0 = 1 + i.$$

$$\int_{C_2} f(z) dz = \int_{C_2} 4y dz = \int_0^1 (1 + 3it^2)4t^3 dt = (t^4 + i2t^6) \Big|_0^1 = 1 + 2i.$$

So

$$\int_C f dz = (1 + i) + (1 + 2i) = 2 + 3i.$$

- (iii) Assume C is parametrized by $z(t), a \leq t \leq b, z(a) = z_1, z(b) = z_2$. Then

$$\int_C 1 dz = \int_a^b z'(t) dt = z(t) \Big|_a^b = z(b) - z(a) = z_2 - z_1.$$

- (iv) We have $z'(\theta) = ie^{i\theta}$, and

$$f(z(\theta)) = e^{(-1+i)(\ln 1 + i\theta)}.$$

So

$$\int_C f(z) dz = \int_0^{2\pi} ie^{i\theta} e^{(-1+i)i\theta} dt = \int_0^{2\pi} ie^{-\theta} dt = -ie^{-\theta} \Big|_0^{2\pi} = i(1 - e^{-2\pi}).$$