

Math 416 Complex variables

Solutions to Problem Set 6

1. Let C be the positively oriented circle $|z| = 1$, with parametric representation $z = e^{i\theta}$ ($0 \leq \theta \leq 2\pi$). and let m and n be integers. Then

$$\int_C z^m \bar{z}^n dz = \int_0^{2\pi} (e^{i\theta})^m (e^{-i\theta})^n i e^{i\theta} d\theta = i \int_0^{2\pi} e^{i(m+1)\theta} e^{-in\theta} d\theta.$$

But we know from Exercise 3, section 41, that

$$\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \begin{cases} 0 & \text{when } m \neq n, \\ 2\pi & \text{when } m = n. \end{cases}$$

Consequently,

$$\int_C z^m \bar{z}^n dz = \begin{cases} 0 & \text{when } m + 1 \neq n, \\ 2\pi i & \text{when } m + 1 = n. \end{cases}$$

2. To find an upper bound for $|\int_C (e^z - \bar{z}) dz|$, we let z be a point on C , and observe that

$$|e^z - \bar{z}| \leq |e^z| + |\bar{z}| = e^x + \sqrt{x^2 + y^2}.$$

But $e^x \leq 1$ since $x \leq 0$, and the distance $\sqrt{x^2 + y^2}$ of the point z from the origin is always less than or equal to 4. Thus $|e^z - \bar{z}| \leq 5$ when z is on C . The length of C is 12. Hence, by writing $M = 5$ and $L = 12$, we have

$$|\int_C (e^z - \bar{z}) dz| \leq ML = 60.$$

3. (a) We know that $|\sin z| \geq |\sin x|$, and on the vertical sides, $|\sin x| = 1$, so $|\sin z| \geq 1$ on the vertical sides.

Also, we know that $|\sin z| \geq |\sinh y|$. We have $\sinh y = \frac{e^y - e^{-y}}{2}$. Since e^y is an increasing function and e^{-y} is a decreasing function, $e^y - e^{-y}$ is an increasing function.

So if $N \geq 0$, $|\sinh((N + \frac{1}{2})\pi)| = \sinh((N + \frac{1}{2})\pi) \geq \sinh \frac{\pi}{2}$. Also, $|\sinh(-(N + \frac{1}{2})\pi)| = |\sinh((N + \frac{1}{2})\pi)| \geq \sinh \frac{\pi}{2}$. We let $A = \max(1, \sinh \frac{\pi}{2})$.

(b) The length of $C_N = 8(N + \frac{1}{2})\pi = 4(2N + 1)\pi$. For every z on C_N , $|z^2| \geq (N + \frac{1}{2})^2\pi^2$. Therefore

$$\left| \frac{1}{z^2 \sin z} \right| \leq \frac{4}{A\pi^2(2N + 1)^2}.$$

So

$$\left| \int_{C_N} \frac{dz}{z^2 \sin z} \right| \leq \frac{4}{A\pi^2(2N + 1)^2} \times 4(2N + 1)\pi = \frac{16}{A\pi(2N + 1)},$$

so the value of the integral tends to zero as N tends to infinity.

4. $\int_i^{i/2} e^{\pi z} dz = \frac{e^{\pi z}}{\pi} \Big|_i^{i/2} = \frac{e^{i\pi/2} - e^{i\pi}}{\pi} = \frac{i+1}{\pi}$.

6. Let C denote any contour from $z = -1$ to $z = 1$ that except for its end points lies above the real axis. This exercise asks us to evaluate the integral

$$\int_{-1}^1 z^i dz,$$

where z^i denotes the principal branch.

An antiderivative of this branch cannot be used since the branch is not even defined at $z = -1$. But the integrand can be replaced by the branch

$$z^i = \exp(i \operatorname{Log} z) \quad (|z| > 0, -\frac{\pi}{2} < \arg z < \frac{3\pi}{2})$$

since it agrees with the integrand along C . Using an antiderivative of this new branch, we can now write

$$\begin{aligned} I &= \frac{z^{i+1}}{i+1} \Big|_{-1}^1 = \frac{1}{i+1} ((1)^{i+1} - (-1)^{i+1}) = \frac{1}{i+1} (e^{(i+1)\log 1} - e^{(i+1)\log(-1)}) \\ &= \frac{1}{i+1} (e^{(i+1)(\ln 1 + i0)} - e^{(i+1)(\ln 1 + i\pi)}) = \frac{1}{i+1} (1 - e^{-\pi} e^{i\pi}) = \frac{1 + e^{-\pi}}{1+i} \cdot \frac{1-i}{1-i} \\ &= \frac{1 + e^{-\pi}}{2} (1-i). \end{aligned}$$