Math 416 Complex variables

Solutions to Problem Set 8

1.

(i) We have

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n},$$

so we get

$$z\cos(z^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{4n+1}.$$

(ii) Since

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n \quad \text{when } |z| < 1$$

we get

$$\frac{z}{z^2+9} = \frac{z}{9} \cdot \frac{1}{1+\frac{z^2}{9}} = \frac{z}{9} \cdot \sum_{n=0}^{\infty} (-1)^n (\frac{z^2}{9})^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{9^{n+1}} z^{2n+1} \quad \text{when } |z| < 3.$$

(iii) Since

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{for } |z| < 1,$$

we write

$$\frac{1}{1-z} = \frac{1}{(1-i)-(z-i)} = \frac{1}{1-i} \cdot \frac{1}{1-\frac{z-i}{1-i}}.$$

So

$$\frac{1}{1-z} = \frac{1}{1-i} \cdot \sum_{n=0}^{\infty} (\frac{z-i}{1-i})^n = \sum_{n=0}^{\infty} \frac{1}{(1-i)^{n+1}} \cdot (z-i)^n$$

when $|\frac{z-i}{1-i}| < 1$, that is when $|z-i| < |1-i| = \sqrt{2}$.

2. Since for evert z, we have

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}$$

we get

$$\cos(\frac{1}{z^2}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{1}{z^{2n}}.$$

Thus

$$\frac{1}{z^2}\cos(\frac{1}{z^2}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{1}{z^{2n+2}} = \frac{1}{z^2} - \frac{1}{2}\cdot\frac{1}{z^4} + \frac{1}{24}\cdot\frac{1}{z^6} + \ldots.$$

3. The function has three singular points 0,1, and -1. So we get two regions on which f is analytic: $D_1 : 0 < |z| < 1$, and $D_2 : |z| > 1$.

On D_1, we have $|z^2| < 1$, so

$$\frac{1}{1-z^2} = \sum_{n=0}^{\infty}(z^2)^n = \sum_{n=0}^{\infty} z^{2n}.$$

So

$$f(z) = \frac{1}{z}\sum_{n=0}^{\infty} z^{2n} = \sum_{n=0}^{\infty} z^{2n-1} = \frac{1}{z} + z + z^3 + \ldots$$

on D_1.

On D_2, we have $|z^2| = |z|^2 > 1$, so $|\frac{1}{z^2}| < 1$, so

$$\frac{1}{1-z^2} = \frac{1}{z^2}\frac{-1}{1-\frac{1}{z^2}} = \frac{-1}{z^2}\sum_{n=0}^{\infty}(\frac{1}{z^2})^n.$$

Therefore, on D_2,

$$f(z) = \frac{1}{z}\cdot\frac{-1}{z^2}\sum_{n=0}^{\infty}(\frac{1}{z^2})^n = \sum_{n=0}^{\infty}\frac{-1}{z^{2n+3}} = -\frac{1}{z^3} - \frac{1}{z^5} - \frac{1}{z^7} + \ldots.$$

4. The summation formula in the example says that

$$\frac{z}{1-z} = \sum_{n=1}^{\infty} z^n \qquad \text{when } |z| < 1.$$

If we put $z = re^{i\theta}$, where $0 < r < 1$, the right hand side becomes:

$$\sum_{n=1}^{\infty}(re^{i\theta})^n = \sum_{n=1}^{\infty} r^n e^{in\theta} = \sum_{n=1}^{\infty} r^n \cos(n\theta) + i\sum_{n=1}^{\infty} r^n \sin(n\theta);$$

and the left hand side takes the form

$$\frac{re^{i\theta}}{1-re^{i\theta}} . \frac{1-re^{-i\theta}}{1-re^{-i\theta}} = \frac{re^{i\theta}-r^2}{1-r(e^{i\theta} - i\theta)+r^2} = \frac{r\cos(\theta) - r^2 + ir\sin(\theta)}{1-2r\cos(\theta)+r^2}.$$

Equating the real parts and the imaginary parts, we get the desired equalities.

5. (a) Since the sequence z_n converges to z, there is n_0 such that for $n > n_0$,

$$|z - z_n| \leq 1.$$

On the other hand, by the triangular inequality,

$$|z_n| \leq |z_n - z| + |z|$$

so $|z_n| \leq 1 + |z|$ for all $n > n_0$. Let $M = \max(|z|+1, |z_1|, |z_2|, \ldots, |z_{n_0}|)$. We get

$$|z| < M \quad \text{for all } n.$$

(b) If we write $z_n = x_n + iy_n$, then if the sequence z_n converges to $z = x + iy$, the sequence x_n converges to x, and the sequence y_n converges to y. Since these two latter sequences are sequences of real numbers, we know from the calculus of real variables that there are two real numbers M_1 and M_2 such that for every integer n, $|x_n| \leq M_1$ and $|y_n| \leq M_2$.

Let $M = \max(M_1, M_2)$. Then

$$|z_n| = \sqrt{|x_n|^2 + |y_n|^2} \leq \sqrt{M_1^2 + M_2^2} \leq \sqrt{2M^2} = M\sqrt{2}.$$

So $|z_n| \leq M\sqrt{2}$ for every n.

6. We have

$$\sin(z) = \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

So we get

$$\cos(z) = -\sin(z - \frac{\pi}{2}) = \sum_{n=0}^{\infty}(-1)^n \frac{-(z-\frac{\pi}{2})^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!} \cdot (z - \frac{\pi}{2})^{2n+1}.$$

7. (a) Let a denote a real number, where $-1 < a < 1$. Since

$$\frac{1}{1-z} = \sum_{z=0}^{\infty} z^n \qquad \text{when } |z| < 1$$

we can write

$$\frac{a}{z-a} = \frac{a}{z}\frac{1}{1-(a/z)} = \sum_{n=0}^{\infty} \frac{a^{n+1}}{z^{n+1}},$$

or

$$\frac{a}{z-a} = \sum_{n=1}^{\infty} \frac{a^n}{z^n} \qquad \text{when } |a| < |z|).$$

(b) Putting $z = re^{i\theta}$ in the above equality, we have

$$\frac{a}{e^{i\theta}-a} = \sum_{n=1}^{\infty} a^n e^{-in\theta}.$$

But

$$\frac{a}{e^{i\theta}-a} = \frac{a}{(\cos(\theta)-a)+i\sin(\theta)} \cdot \frac{(\cos(\theta)-a)-i\sin(\theta)}{(\cos(\theta)-a)-i\sin(\theta)} = \frac{a\cos(\theta)-a^2-ia\sin\theta}{1-2a\cos\theta+a^2}.$$

and

$$\sum_{n=1}^{\infty} a^n e^{-in\theta} = \sum_{n=1}^{\infty} a^n \cos(n\theta) - i\sum_{n=1}^{\infty} a^n \sin(n\theta).$$

Comparing the real parts and the imaginary parts, we get the desired result.