Math 416 Complex variables

Solutions to Problem Set 9

1. You can show that the given function is entire by finding the Laurent series of $f(z)$ at $z = \pm\pi/2$.

You can also argue as follows: First we show that the function

$$g(z) = \begin{cases} \frac{\sin z}{z} & \text{when } z \neq 0 \\ 1 & \text{when } z = 0 \end{cases}$$

is entire. This is because the Laurent series of $g(z)$ around the origin is given by

$$g(z) = \frac{1}{z}\sum_{n=0}^{\infty}(-1)\frac{z^{2n+1}}{(2n+1)!} = \frac{1}{z}(z - \frac{z^3}{3!} + \dots) = 1 - \frac{z^2}{3!} + \dots.$$

which is valid for every z with $|z| > 0$. But since for $z = 0$, the value of the Taylor series agrees with the value of $g(z)$ too, we can conclude that $g(z)$ is an entire function.

Next, to show that $f(z)$ is entire, we need to show that $f(z)$ is analytic at $\pm\pi/2$. We show here that $f(z)$ is analytic at $z = \pi/2$. The case of $z = -\pi/2$ is similar.

Define a function $h(z) = f(z + \pi/2)$. To show that $f(z)$ is analytic at $\pi/2$, it is enough to show that $h(z)$ is analytic at 0. But we have

$$h(z) = f(z + \pi/2) = \begin{cases} \frac{-\sin z}{z(z+\pi)} & \text{when } z \neq 0, -\pi \\ -\frac{1}{\pi} & \text{when } z = 0, -\pi \end{cases}$$

And so $h(z) = \frac{-g(z)}{z+\pi}$ for $z \neq -\pi$. Since $g(z)$ and $z + \pi$ are analytic at 0, and the denominator is non-zero at the origin, we can conclude that $h(z)$ is analytic at 0.

4. We are given that $f(z)$ is an entire function represented by a series of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

(a) Write $g(z) = f(f(z))$ and observe that

$$f(f(z)) = g(0) + \frac{g'(0)}{1!}z + \frac{g''(0)}{2!}z^2 + \frac{g^{(3)}}{3!}z^3 + \dots$$

It is straightforward to show that

$$g'(z) = f'(f(z))f'(z),$$

$$g''(z) = f''(f(z))[f'(z)]^2 + f'(f(z))f''(z),$$

and

$$g^{(3)}(z) = f^{(3)}(z)[f'(z)]^3 + 2f'(z)f''(z)f''(f(z)) + f''(f(z))f'(z)f''(z) + f'(f(z))f^{3)}(z).$$

Thus

$$g(0) = 0, \quad g'(0) = 1, \quad g''(0) = 4a_2, \quad g^{(3)}(0) = 12(a_2^2 + a_3),$$

and so

$$f(f(z)) = z + 2a_2z^2 + 2(a_2^2 + a_3)z^3 + \dots$$

(b) We have

$$\begin{aligned} f(f(z)) &= f(z) + a_2[f(z)]^2 + a_3[f(z)]^3 + \dots \\ &= (z + a_2z^2 + a_3z^3 + \dots) + a_2(z + a_2z^2 + a_3z^3 + \dots)^2 + a_3(z + a_2z^2 + \dots)^3 + \dots \\ &= (z + a_2z^2 + a_3z^3 + \dots) + (a_2z^2 + 2a_2^2z^3 + \dots) + (a_3z^3 + \dots) \\ &= z + 2a_2z^2 + 2(a_2^2 + a_3)z^3 + \dots . \end{aligned}$$

(c) Since

$$\sin z = z - \frac{z^3}{3!} + \cdots = z + 0\, z^2 + (-\frac{1}{6})z^3 + \dots$$

the result in part (a), with $a_2 = 0$ and $a_3 = -\frac{1}{6}$, tells us that

$$\sin(\sin z) = z - \frac{1}{3}z^3 + \dots .$$

5. (b) Since

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

we have

$$e^{-z} = \frac{e^{-(z-1)}}{e} = \frac{1}{e}(1 - (z-1) + \frac{(z-1)^2}{2} - \frac{(z-1)^3}{6} + \dots).$$

Therefore,

$$\frac{e^{-z}}{(z-1)^2} = \frac{1}{e}(\frac{1}{(z-1)^2} - \frac{1}{z-1} + \frac{1}{2} - \dots),$$

so the residue of $f(z) = \frac{e^{-z}}{(z-1)^2}$ at $z = 1$ is $-\frac{1}{e}$. The Cauchy's integral formula now tells us that

$$\int_C \frac{e^{-z}}{(z-1)^2}\, dz = 2\pi i(-\frac{1}{e}) = -\frac{2\pi i}{e}.$$

(d) We need the two residues of $\frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)}$ at $z = 0$ and $z = 2$. The residue at $z = 0$ can be found by writing

$$\begin{aligned}\frac{z+1}{z(z-2)} &= (\frac{z+1}{z})(\frac{1}{z-2}) = (-\frac{1}{2})(1+\frac{1}{z}) \cdot \frac{1}{1-(z/2)} \\ &= (-\frac{1}{2} - \frac{1}{2} \cdot \frac{1}{z})(1 + \frac{z}{2} + \frac{z^2}{2^2} + \dots),\end{aligned}$$

which is valid when $0 < |z| < 2$. So the residue is $-\frac{1}{2}$.

To obtain the residue at $z = 2$, we write

$$\begin{aligned}\frac{z+1}{z(z-2)} &= \frac{(z-2)+3}{z-2} \cdot \frac{1}{2+(z-2)} = \frac{1}{2}(1+\frac{3}{z-2}) \cdot \frac{1}{1+(z-2)/2} \\ &= \frac{1}{2}(1+\frac{3}{z-2})(1 - \frac{z-2}{2} + \frac{(z-2)^2}{2^2} - \dots),\end{aligned}$$

which is valid when $0 < |z-2| < 2$, so the residue is $\frac{3}{2}$. Finally,

$$\int_C \frac{z+1}{z^2-2z}\, dz = 2\pi i(-\frac{1}{2} + \frac{3}{2}) = 2\pi i.$$

6. (a) If $f(z) = \frac{z^5}{1-z^3}$, then

$$\frac{1}{z^2} f(\frac{1}{z}) = \frac{1}{z^7 - z^4} = -\frac{1}{z^4} \cdot \frac{1}{1-z^3} = -\frac{1}{z^4}(1 + z^3 + z^6 + \dots) = -\frac{1}{z^4} - \frac{1}{z} - z^2 - \dots$$

when $0 < |z| < 1$. This tells us that

$$\int_C f(z)\, dz = 2\pi i \operatorname{Res}_{z=0} \frac{1}{z^2} f(\frac{1}{z}) = 2\pi i(-1) = -2\pi i.$$

(b) When $f(z) = \frac{1}{1+z^2}$, we have

$$\frac{1}{z^2} f(\frac{1}{z}) = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = 1 - z^2 + z^4 - \dots$$

for $0 < |z| < 1$. Thus

$$\int_C f(z)\, dz = 2\pi i \operatorname{Res}_{z=0} \frac{1}{z^2} f(\frac{1}{z}) = 2\pi i(0) = 0.$$

7. Let $f(z) = \frac{P(z)}{Q(z)}$. Then

$$\begin{aligned} f(\frac{1}{z}) = \frac{P(\frac{1}{z})}{Q(\frac{1}{z})} &= \frac{a_0 + a_1 \frac{1}{z} + \cdots + a_n \frac{1}{z^n}}{b_0 + b_1 \frac{1}{z} + \cdots + b_m \frac{1}{z^m}} \\ &= \frac{a_n z^{m-n} + a_{n-1} z^{m-n+1} + \cdots + a_1 z^{m-1} + a_0 z^m}{b_m + b_{m-1} z + \cdots + b_1 z^{m-1} + b_0 z^m}. \\ &= (\frac{a_n}{b_m} z^{m-n} + \frac{a_{n-1} - b_{m-1} \frac{a_n}{b_m}}{b_m} z^{m-n+1} + \dots). \end{aligned}$$

So

$$\frac{1}{z^2} f(\frac{1}{z}) = (\frac{a_n}{b_m} z^{m-n-2} + \frac{a_{n-1} - b_{m-1} \frac{a_n}{b_m}}{b_m} z^{m-n-1} + \dots)$$

Since $m - n - 2 \geq 0$, all the powers of z appearing in the Laurent series of $\frac{1}{z^2} f(\frac{1}{z})$ are non-negative, and therefore, the residue at $z = 0$ is 0. So the given integral is also zero.