2. (a) For which real number k is the following matrix invertible?

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ -1 & k & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 9 \\ 0 & 0 & 1 & 4 & 16 \\ 0 & 0 & 1 & k & k^2 \end{pmatrix}$$

[5 points]

$$\det(A) = \det\begin{pmatrix} 1 & 1 \\ -1 & K \end{pmatrix} \det\begin{pmatrix} 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & K & K^2 \end{pmatrix}$$

$$= (K+1)(K^2 - 7K + 12)$$

$$= (K+1)(K-3)(K-4)$$

Therefore, $\det(A) = 0 \Leftrightarrow K = 3, 4, -1$

so A is invertible for every $K \neq 3, 4, -1$.

(b) Suppose

$$\underset{A}{\begin{pmatrix} 4 & 1 & 10 \\ 2 & -2 & 17 \\ -2 & 4 & 14 \end{pmatrix}} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}.$$

Use **Cramer's rule** to find x_2.

[3 points]

$B_2 = \begin{pmatrix} 4 & 2 & 10 \\ 2 & 1 & 17 \\ -2 & -1 & 14 \end{pmatrix}$. Since the second column of B_2 is a multiple of its first column, $\det B_2 = 0$

We have $x_2 = \dfrac{\det(B_2)}{\det(A)_3} = 0$.

3. Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 \\ 3 & 1 \end{pmatrix}$. Determine the matrix of the linear transformation

$$T : F^{2\times 2} \to F^{2\times 2}$$

defined by

$$T(M) = AMB$$

with respect to the basis $\mathcal{B} = \{E^{1,1}, E^{1,2}, E^{2,1}, E^{2,2}\}$.
($E^{i,j}$ is the matrix which has 1 in the ij-th entry and 0 elsewhere). [6 points]

We have

$$T(E^{1,1}) = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$T(E^{1,2}) = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix}$$

$$T(E^{2,1}) = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -2 & 0 \end{pmatrix}$$

$$T(E^{2,2}) = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$$

Therefore

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ -1 & 3 & -2 & 6 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

4. The *dot product* of two vectors α and β in $F^{n\times 1}$ is defined as follows: if $\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\beta = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$, then

$$\alpha \cdot \beta = a_1 b_1 + \cdots + a_n b_n.$$

If A is a matrix with column vectors $\alpha_1, \ldots, \alpha_n$, then show that

$$(\det A)^2 = \det \begin{pmatrix} \alpha_1 \cdot \alpha_1 & \alpha_1 \cdot \alpha_2 & \cdots & \alpha_1 \cdot \alpha_n \\ \alpha_2 \cdot \alpha_1 & \alpha_2 \cdot \alpha_2 & \cdots & \alpha_2 \cdot \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n \cdot \alpha_1 & \alpha_n \cdot \alpha_2 & \cdots & \alpha_n \cdot \alpha_n \end{pmatrix}$$

[5 points]

We have $\alpha \cdot \beta = \alpha^T \beta$.

So

$$\det \begin{pmatrix} \alpha_1 \cdot \alpha_1 & \cdots & \alpha_1 \cdot \alpha_n \\ \vdots & & \vdots \\ \alpha_n \cdot \alpha_1 & \cdots & \alpha_n \cdot \alpha_n \end{pmatrix} = \det \left[\begin{pmatrix} \alpha_1^t \\ \vdots \\ \alpha_n^T \end{pmatrix} \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix} \right]$$

$$= \det(A^t A) = \det(A^t) \det(A) = \det(A)\det(A) = \det(A)^2.$$

5. Label the following statements as TRUE or FALSE, giving a short explanation (e.g. a proof or counterexample). [13 points]

(a) If W is a proper subspace of a finite-dimensional vector space V (i.e. $W \neq V$), then there is a non-zero linear functional $f \in V^*$ such that f is zero on each vector in W.

True: since $W \neq V$, $W^0 \neq V^0 = \{0\}$

$\Rightarrow$ there is $0 \neq f \in W^0 \subset V^*$.

(b) The function $D : F^{3\times 3} \to F$ define by

$$D\left(\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\right) = a_{12}a_{21}a_{23}$$

is a 3-linear function.

False. $D\left(\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}\right) + D\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3' \end{pmatrix}\right)$

$= a_{11}a_{21}a_{23} + a_{11}a_{21}a_{23} = 2a_{11}a_{21}a_{23}$

$\neq D\left(\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 + \alpha_3' \end{pmatrix}\right) = a_{11}a_{21}a_{23}$

(c) If $T : V \to W$ is a linear transformation between finite-dimensional vector spaces and T^t denote the transpose of T, then T is one-one if and only if T^t is onto.

True: T is one-one $\Leftrightarrow$ nullity $(T) = 0$

$\Leftrightarrow$ rank $(T) = \dim V$

$\Leftrightarrow$ rank $(T^t) = \dim V$

$\Leftrightarrow$ T^t is onto.

(d) If $\mathcal{B} = \{\epsilon_1, \epsilon_2, \epsilon_3\}$ is the standard basis for $\mathbf{R}^3$ and $\mathcal{B}' = \{\epsilon_1, \epsilon_2 - \epsilon_1, \epsilon_3 - \epsilon_2\}$, then the matrix of change of coordinates from $\mathcal{B}'$ to $\mathcal{B}$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

False: The matrix of change of coordinates from $\mathcal{B}'$ to $\mathcal{B}$ is

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}.$$

(e) If V is a 3-dimensional vector space, $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ a basis for V, and f_1, f_2, f_3 the corresponding dual basis for V^*, then

$$f_1, 2f_2, 3f_3$$

is the dual basis corresponding to the basis

$$\mathcal{B}' = \{\alpha_1, 2\alpha_2, 3\alpha_3\}.$$

False: $2f_2\,(2\alpha_2) = 4\,f_2(\alpha_2) = 4 \neq 1.$