Solutions to the selected problems (Homework 10)

Linear Algebra

Fall 2010

Page 198, 7) $\mathcal{B} = \{1, x, x^2, \dots, x^n\}$ is a basis for V. The matrix of D with respect to $\mathcal{B}$ is lower triangular, and all the diagonal entries are zero, so the characteristic polynomial of D is x^{n+1}. So the minimal polynomial of D should be of the form x^k for some $1 \leq k \leq n+1$. Since $D^n(x^n) = D \circ D \circ \dots \circ D(x^n) = n(n-1)\dots 1$, D^n is not the zero linear transformation, so if $p(x) = x^n$, $p(D) \neq 0$, so the minimal polynomial should be x^{n+1}.

Page 205, 3) Since for every $\alpha \in W$, $T_W(\alpha) = T(\alpha) = c\alpha$, T_W is simply c times the identity linear transformation.

Page 205, 4) The characteristic polynomial of A is x^3, therefore, Theorem 5 of page 203 shows that A is similar to an upper triangular matrix. The nullspace $\{X | AX = 0\}$ is 1-dimensional and it is spanned by the vector $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. If A is similar to an upper triangular matrix U, then since the only characteristic value of A is zero, all the entries on the diagonal of U should be zero (since U and A have the same characteristic values). So assume U has the form

$$\begin{pmatrix} 0 & c_1 & c_2 \\ 0 & 0 & c_3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose that U corresponds to the basis $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$. Then

$$A\alpha_1 = 0, A\alpha_2 = c_1\alpha_1, A\alpha_3 = c_2\alpha_1 + c_3\alpha_2.$$

Since $\alpha_1 \in \{X|AX = 0\}$, we can assume $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. We have $A\alpha_2 = c_1\alpha_1$. Since the null space $\{X|AX = 0\}$ is 1-dimensional, $c_1 \neq 0$, and $A\frac{\alpha_2}{c_1} = \alpha_1$. Since if $\alpha_1, \alpha_2, \alpha_3$ form a basis for $F^{3\times 1}$, then $\alpha_1, \frac{\alpha_2}{c_1}, \alpha_3$ form a basis as well, we can assume that $c_1 = 1$ from the beginning. So we solve the equation $AX = \alpha_1$: we get $X = \begin{pmatrix} x_1 \\ 1 \\ 1 - x_1 \end{pmatrix}$. So we can let $\alpha_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$. Now α_3 could be any vector such that $\alpha_1, \alpha_2, \alpha_3$ is linearly independent. For example, let $\alpha_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then $A\alpha_3 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = 2\alpha_2$, so

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$$

(there are many other possible U.)

Page 205, 5) If $p(x) = x(x-1)$, then $p(A) = 0$. So $m_A(x)$ should divide $p(x)$ (since otherwise the remainder of $p(x)$ by $m_A(x)$ would be a polynomial whose degree is smaller than $m_A(x)$ and vanishes at A). So there are three possibilities for $m_A(x)$: x, $x-1$, or $x(x-1)$. If $m_A(x) = x$, then $A = 0$. If $m_A(x) = x - 1$, then $A = I$, and if $m_A(x) = x(x-1)$, then Theorem 6 of page 204 tells us that A is similar to a diagonal matrix.

Page 213, 2) Suppose that $\dim V = n$. Let $\{\alpha_1^1, \ldots, \alpha_{m_1}^1\}$ be a basis for W_1, $\{\alpha_1^2, \ldots, \alpha_{m_2}^2\}$ a basis for W_2, ..., $\{\alpha_1^k, \ldots, \alpha_{m_k}^k\}$ a basis for W_k. Then we have $n = m_1 + \cdots + m_k$ by our assumption. Since the n vectors $\alpha_1^1, \ldots, \alpha_{m_1}^1, \ldots, \alpha_{m_k}^k$ span V (Since $V = W_1 + \cdots + W_k$, every vector $\beta \in V$ is of the form $\beta_1 + \cdots + \beta_k$ where $\beta_i \in W_i$), they should be linearly independent, so they form a basis for V.

Now if we have $\gamma_1 + \cdots + \gamma_k = 0$ for some vectors $\gamma_i \in W_i$, then we can write $\gamma_1 = c_1^1\alpha_1^1 + \ldots c_{m_1}^1\alpha_{m_1}^1, \ldots, \gamma_k = c_1^k\alpha_1^k + \cdots + c_{m_k}^k\alpha_{m_k}^k$, so we see that a linear combination of the α_j^i is zero, so all the coefficients should be zero, so $c_j^i = 0$ for every i and j, hence each γ_i is zero. Therefore $W_1, \ldots, W_k$ are independent.

Page 225 1). The characteristic polynomial of the matrix is

$$x^3 - 2x^2 + x - 2 = (x-2)(x^2+1),$$

Since $A - 2I \neq 0$ and $A^2 + I \neq 0$, the minimal polynomial is also $(x-2)(x^2+1)$, so $p_1 = x - 2$ and $p_2 = x^2 + 1$.

The nullspace of $p_1(T)$ is the nullspace of $T - 2I$ which is spanned by $\alpha_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. The nullspace of $p_2(T)$ is the nullspace of

$$\begin{pmatrix} 5 & -5 & 0 \\ 0 & 0 & 0 \\ 10 & -10 & 0 \end{pmatrix}$$

which is spanned by $\alpha_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\alpha_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Since $T(\alpha_1) = 2\alpha_1$, the matrix of the restriction of T to W_1 is 1 by 1 and equal to [2]. The matrix of the restriction of T to W_2 is 2 by 2. Since $T(\alpha_2) = \begin{pmatrix} -2 \\ -2 \\ -3 \end{pmatrix} = -3\alpha_2 - 2\alpha_3$, and $T(\alpha_3) = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} = 5\alpha_2 + 3\alpha_3$, the matrix of the restriction of T to W_2 with respect to the basis $\{\alpha_2, \alpha_3\}$ is

$$\begin{pmatrix} -3 & 5 \\ -2 & 3 \end{pmatrix}.$$