# Solutions to the selected problems (Homework 11)

Linear Algebra

Fall 2010

noindent Page 275, 2). If $(\alpha|\beta)$ and $<\alpha, \beta>$ are two inner product, then the sum and the difference of them is linear with respect to the first component, and

$$(\beta|\alpha) \pm <\beta|\alpha> = \overline{(\alpha|\beta)} \pm \overline{<\alpha|\beta>} = \overline{(\alpha|\beta) \pm <\alpha|\beta>}$$

So it remain the last property: if $(\alpha|\alpha) + <\alpha|\alpha> = 0$, then since the sum of two non-negative numbers is zero, each should be equal to zero, so $(\alpha|\alpha) = 0$, so $\alpha = 0$. Therefore the sum of two inner products is an inner product. The difference of two inner products is not necessarily an inner product. For example, if $<\alpha|\alpha> = (\alpha|\alpha)$ (so the two inner products are equal), their difference is zero for every pair which clearly does not satisfy the last property of an inner product.

Page 276, 8) $f_A$ is clearly linear with respect to both $X$ and $Y$:

$$\begin{aligned} f_A(X, Y_1 + cY_2) = (Y_1 + cY_2)^t AX = (Y_1^t + c\, Y_2^t)AX &= Y_1^t AX + c\, Y_2^t AX \\ &= f_A(X, Y_1) + c f_A(X, Y_2). \end{aligned}$$

A similar argument shows that $f_A$ is linear with respect to $X$.

If $f_A$ is an inner product, we should have $f_A(X, Y) = f_A(Y, X)$ and $f_A(X, X) > 0$. Suppose that

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let

$$\epsilon_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \epsilon_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Then we have $f_A(\epsilon_1, \epsilon_2) = c$ and $f_A(\epsilon_2, \epsilon_1) = b$, so $b = c$. Also $f_A(\epsilon_1, \epsilon_1) = a$, $f_A(\epsilon_2, \epsilon_2) = d$, so $a, d > 0$. Let $a' = \sqrt{a}$. Now for $0 \neq X = \begin{pmatrix} x \\ y \end{pmatrix}$, we have

$$0 < f_A(X, X) = ax^2 + 2bxy + dy^2 = (a'x + \frac{b}{a'}y)^2 + (d - \frac{b^2}{a})y^2.$$

If we let $x = -\frac{b}{(a')^2}$, $y = 1$, then

$$f_A(X, X) = d - \frac{b^2}{a}$$

which should be positive, so $\det A > 0$.

Conversely, if $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$ is symmetric, and $a, d, \det A > 0$, then for $0 \neq X = \begin{pmatrix} x \\ y \end{pmatrix}$,

$$f_A(X, X) = ax^2 + 2bxy + dy^2 = (a'x + \frac{b}{a'}y)^2 + (d - \frac{b^2}{a})y^2.$$

This is zero only if $y = 0$, $x = -\frac{b}{(a')^2}y = 0$. Therefore $f_A$ is an inner product.

Page 289, 5) One direction is clear. For the other direction, if $(\alpha, \gamma) = (\beta|\gamma)$ for every $\gamma$, then $(\alpha - \beta|\gamma) = 0$ for every $\gamma$, in particular,

$$(\alpha - \beta|\alpha - \beta) = 0.$$

Hence $\alpha - \beta = 0$.

Page 289, 9) (a) We are looking for polynomials $f(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ such that $(c|f) = \int_0^1 cf(x)\, dx = 0$ for every real number $c$. So $c(\frac{a_3}{4} + \frac{a_2}{3} + \frac{a_1}{2} + a_0) = 0$ for every $c$. Therefore $\frac{a_3}{4} + \frac{a_2}{3} + \frac{a_1}{2} + a_0 = 0$. So $3a_3 + 4a_2 + 6a_1 + 12a_0 = 0$. A basis for this subspace is $\{-4x^3 + 1, -3x^2 + 1, -2x + 1\}$.

(b) We have

- $\alpha_1 = 1$.
- $||\alpha_1|| = 1$, $(x|\alpha_1) = \int_0^1 x\, dx = \frac{1}{2}$, so

$$\alpha_2 = x - \frac{1}{2}.$$

- $(x^2|\alpha_1) = \frac{1}{3}$, $(x^2|\alpha_2) = \int_0^1 x^3 - \frac{x^2}{2}\ dx = \frac{1}{12}$, $||\alpha_2||^2 = (\alpha_2|\alpha_2) = \int_0^1 (x - \frac{1}{2})^2\ dx = \frac{1}{12}$. Therefore,

$$\alpha_3 = x^2 - \frac{1}{3} - \alpha_2 = x^2 - x + \frac{1}{6}.$$

- $(x^3|1) = \frac{1}{4}, (x^3|\alpha_2) = \frac{3}{40}, (x^3|\alpha_3) = \frac{1}{120}, (\alpha_3|\alpha_3) = \int_0^1 (x^2 - x + \frac{1}{6})^2\ dx = \frac{1}{180}$. Therefore

$$\begin{aligned}\alpha_4 &= x^3 - \frac{1}{4} - \frac{9}{10}(x - \frac{1}{2}) - \frac{3}{2}(x^2 - x + \frac{1}{6}) \\ &= x^3 - \frac{3}{2}x^2 + \frac{3}{5}x - \frac{1}{20}.\end{aligned}$$