Solutions to the selected problems (Homework 12)

Linear Algebra

Fall 2010

Page 289, 6) (a) Let $\alpha = (3,4)$. By Theorem 4 of page 284, $E(\beta)$ which is the same as the best approximation to β in W, is given by $E(\beta) = \frac{(\beta|\alpha)}{||\alpha||^2}\alpha$. Since $||\alpha|| = 5$, if $\beta = (x_1, x_2)$, we have

$$E(\beta) = E(x_1, x_2) = \frac{3x_1 + 4x_2)}{25}(3,4).$$

(b) Since $E(1,0) = \frac{3}{25}(3,4) = (\frac{9}{25}, \frac{12}{25})$, and $E(0,1) = \frac{4}{25}(3,4) = (\frac{12}{25}, \frac{16}{25})$, the matrix is

$$\frac{1}{25}\begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix}.$$

(c) $W^{\perp}$ is 1-dimensional and is spanned by any vector orthogonal to $(3,4)$, so $W^{\perp}$ is spanned by $(-4,3)$.

(d) We are looking for orthonormal vectors β_1, β_2 such that $E(\beta_1) = \beta_1$, and $E(\beta_2) = 0$, so $\beta_1 \in W$ and $\beta_2 \in W^{\perp}$. Since the norms of β_1 and β_2 are both 1, we have $\beta_1 = \frac{(3,4)}{||(3,4)||} = (\frac{3}{5}, \frac{4}{5})$, $\beta_2 = \frac{(-4,3)}{||(-4,3)||} = (\frac{-4}{5}, \frac{3}{5})$. (we could also choose β_1 and/or $-\beta_2$.)

Page 290, 12) If we write α as $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$ where $\alpha_1, \beta_1 \in W$, $\alpha_2, \beta_2 \in W^{\perp}$, then $E(\alpha) = \alpha_1$ and $E(\beta) = \beta_1$. Therefore,

$$\begin{aligned}
(E(\alpha)|\beta) &= (\alpha_1|\beta_1 + \beta_2) \\
&= (\alpha_1|\beta_1) + (\alpha_1|\beta_2) \\
&= (\alpha_1|\beta_1) \\
&= (\alpha_1 + \alpha_2|\beta_1) \quad \text{since } \beta_1 \in W, \alpha_2 \in W^{\perp}, (\alpha_2|\beta_1) = 0 \\
&= (\alpha|\beta_1) \\
&= (\alpha|E(\beta)).
\end{aligned}$$

Page 290, 13) Note that if W is the subspace spanned by S, then $W^\perp = S^\perp$. The reason is that since $S \subset W$, $W^\perp \subset S^\perp$, and on the other hand if $\alpha \in S^\perp$ and $\beta \in W$, then β can be written as

$$\beta = c_1\beta_1 + \cdots + c_m\beta_m$$

where $\beta_1, \ldots, \beta_m \in S$ and $c_1, \ldots, c_m$ are scalars. So

$$(\alpha|\beta) = c_1(\alpha|\beta_1) + \cdots + c_m(\alpha|\beta_m) = 0,$$

thus $\alpha \in W^\perp$. So $S^\perp \subset W^\perp$, therefore $S^\perp = W^\perp$.

If $\alpha \in S$, then for every $\beta \in S^\perp$, $(\alpha|\beta) = 0$, so by definition, $\alpha \in (S^\perp)^\perp$. Now suppose that V is of dimension n, and W is of dimension m. Then $W^\perp = S^\perp$ by the above argument, and therefore $(W^\perp)^\perp = (S^\perp)^\perp$. We have

$$V = W \oplus W^\perp,$$

therefore $\dim W^\perp = \dim V - \dim W = n - m$. We also have

$$V = (W^\perp) + (W^\perp)^\perp,$$

therefore $\dim (W^\perp)^\perp = \dim V - \dim W^\perp = n - (n - m) = m$. We have shown that $W \subset (W^\perp)^\perp$ for every subset W of V, and since $\dim W = \dim (W^\perp)^\perp$, we get $W = (W^\perp)^\perp = (S^\perp)^\perp$.