Solutions to the selected problems (Homework 8,9)

Linear Algebra

Fall 2010

Page 163, 8) Since $T_B(I) = IB - BI = 0$, the nullspace of T_B is not the zero space, therefore T_B is singular and not invertible, so $\det T_B = 0$.

Page 190, 8) To show that $I - BA$ is invertible, it is enough to show that

$$(I - BA)(I + B(I - AB)^{-1}A) = I.$$

(then we can conclude that $(I+B(I-AB)^{-1}A)(I-BA) = I$, and so $I-BA$ is invertible). This is shown as follows:

$$\begin{aligned}(I - BA)(I + B(I - AB)^{-1}A) &= I - BA + B(I - AB)^{-1}A - BAB(I - AB)^{-1}A \\ &= I - BA + B(I - AB)(I - AB)^{-1})A \\ &= I - BA + BA \\ &= I.\end{aligned}$$

Page 190, 6) The matrix is lower triangular with all the diagonal entries zero, so the only characteristic value for the matrix is zero. If it is diagonalizable, then the nullspace:

$$\{X|AX = 0\}$$

should be four dimensional, so every vector in $\mathbf{R^{4\times 1}}$ should be a in the above set which means $A = 0$. So A is diagonalizable if and only if $a = b = c = 0$.

Page 190, 7) This is an immediate corollary of the lemma on page 186. Clearly, if c_i is a characteristic value of T, then the dimension of the corresponding space of characteristic vectors:

$$W_i := \{X|T(X) = c_iX\}$$

is at least 1. By our assumption,

$$c_T(x) = (x - c_1) \dots (x - c_n)$$

where the c_i are distinct. So if $W := W_1 + \cdots + W_n$, then

$$\dim W = \dim W_1 + \cdots + \dim W_n \geq n,$$

but W is a subspace of V, so $\dim W$ should be equal to n, and $\dim W_i = 1$ for each i. Now if we choose one characteristic vector for each c_i, we get a basis of V consisting of characteristic vectors by the lemma. The matrix of T with respect to this basis is diagonal.

Page 190, 9) Assume c is a characteristic value for AB. We show it is also a characteristic value for BA. We consider two different cases:

1) $c = 0$: then to say 0 is a characteristic value for AB is to say that AB is not invertible. then by the above problem, BA is also not invertible, so 0 i a characteristic value for BA as well.

2) $c \neq 0$: then $cI - AB$ is not invertible, so $c\left(I - \frac{1}{c}(AB)\right)$ is not invertible. Therefore $I - \frac{1}{c}(AB)$ is not invertible, so $I - (\frac{1}{c}A)B$ is not invertible, so by the above problem, $I - B(\frac{1}{c}A)$ is not invertible either. Thus $\frac{1}{c}(cI - BA)$ is not invertible, so $cI - BA$ is not invertible and c is a characteristic value for BA.

Page 190, 10), Let $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$. Then

$$c_A(x) = \det(xI - A) = (x - a)(x - d) - b^2 = x^2 - (a + d)\,x + (ad - b^2).$$

The roots of this polynomial are

$$\frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - b^2)}}{2}.$$

Since $(a + d)^2 - 4(ad - b^2) = (a - d)^2 + 4b^2 > 0$ unless $a = d$ and $b = 0$, we see that if $a \neq d$ or $b \neq 0$, the polynomial has two distinct real roots, so by Problem 7 of page 190, the matrix is diagonalizable. If $a = d$ and $b = 0$, then A is already in the diagonal form, so there is nothing to prove.

Page 190, 13) Assume that there is a continues function f such that

$$\int_0^x f(t)\, dt = c\, f(x),$$

for a real number c. We show that $f = 0$. To show this, we first show that $f(x) = 0$ for every $x \geq 0$. An identical argument shows that $f(x) = 0$ for every $x \leq 0$. Note that $f(0) = 0$ by the above equality. Let a be the largest non-negative real number such that f is zero on the interval $(0, a]$ (if there is no such largest a, then f is zero everywhere on $(0, \infty)$. Note that a could be equal to zero). Then we can choose a positive real number $b > a$ such that f is no where zero on the open interval (a, b). Since f is continuous, f should be everywhere positive, or everywhere negative on the open interval (a, b). We can assume without loss of generality that f is positive for every $a < x < b$.

Since the left hand side of the above equation is differentiable and its derivative is $f(x)$ (by the fundamental theorem of calculus), the right hand side is also differentiable, and we have

$$f(x) = cf'(x).$$

Then we have

$$\frac{f'(x)}{f(x)} = \frac{1}{c},$$

therefore

$$(\ln f(x))' = \frac{1}{c},$$

so $\ln(f)$ is a linear function, i.e. it is of the form $\ln(f(x)) = \frac{1}{c}x + r$ for a constant r. Therefore, $f(x) = e^{\frac{1}{c}x+r}$ on the interval (a, b). Since f is continuous $f(a) = e^{\frac{1}{c}a+r}$, but $f(a) = 0$, so this is impossible.

Page 205, 1 (a): Any one-dimensional invariant subspace is spanned by a non-zero characteristic vector. But

$$c_A(x) = (x - 1)(x - 2) + 2 = x^2 - 3x + 4.$$

The roots of this polynomial are

$$\frac{3 \pm \sqrt{-7}}{2}$$

which are not real numbers. So A does not have any characteristic value which is real number. Therefore over $\mathbf{R}$, the matrix does not have any non-zero characteristic vector, and thus every invariant subspace is either zero dimensional or 2-dimensional.