Complex Analysis, Fall 2017

Problem Set 1

Due: September 12 in class

1. Find $\arg(\frac{i}{-2-2i})$.

2. Find all the points at which $f(z) = \bar{z}^2 + z$ is complex differentiable.

3. Find the following roots.

   (i) All the fourth roots of $-1 - \sqrt{3}\, i$. Express the roots in rectangular coordinates and exhibit them as vertices of a certain square.

   (ii) The square roots of $8i$.

4. If $z_1, z_2 \in \mathbf{C}$, then show that $z_1 \bar{z}_2 = -1$ if and only if the pre-image of z_1 and z_2 under the stereographic projection correspond to diametrically opposite points on the Riemann sphere.

5. Let U_1 and U_2 be open subset of $\mathbf{C}$ and $f : U_1 \to U_2$ and $g : U_2 \to \mathbf{C}$ functions which are real differentiable. Prove the following chain rule:

$$\frac{\partial (f \circ g)}{\partial z} = \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{g}}{\partial z}.$$

(If $f = u + iv$, then $\bar{f} = u - iv$). Similarly

$$\frac{\partial (f \circ g)}{\partial \bar{z}} = \frac{\partial f}{\partial z}\frac{\partial g}{\partial \bar{z}} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{g}}{\partial \bar{z}}.$$

(You don't need to prove the second identity.)

6. Let $U \subset \mathbf{C}$ be a connected open subset and $f : U \to \mathbf{C}$ a holomorphic function.

   (a) Show that if $Re(f)$ is constant on U, then f is constant on U.

(b) Show that if $|f|$ is constant on U, then f is constant on U. (Hint: compute $\frac{\partial(f\bar{f})}{\partial z}$.)

7. Assume f is a polynomial with complex coefficients. Prove that if a root of f' belongs to the boundary of $\Delta(f)$, then either f has a repeated root on that boundary or all the zeros of f lie on a line. (a root z_0 of f is called a repeated root if $(z - z_0)^2$ is a factor of f.)

8. Let $\{z_n\}$ be a sequence of complex numbers and $\lim_{n\to\infty} |\frac{a_n}{a_{n+1}}| = B$ (B could be ∞). Show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ is B.