Complex Analysis, Fall 2017

Problem Set 10

Due: December 8

1. Let $\mathbf{D}$ be the unit disk and S^1 the unit circle.

   (a) Show that if $g : \overline{\mathbf{D}} \to \mathbf{C}$ is a continuous function and $g_r : S^1 \to \mathbf{C}$ is defined by $g_r(z) = g(rz)$, then $g_r(z) \to g(z)$ uniformly for $z \in S^1$ as $r \to 1^-$.

   (b) If $f : S^1 \to \mathbf{C}$ is a continuous function, define $\tilde{f} : \overline{\mathbf{D}} \to \mathbf{C}$ by $\tilde{f}(z) = f(z)$ for $z \in S^1$ and
   $$\tilde{f}(re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) P_r(\theta - \phi)\, d\theta.$$
   (So the real and imaginary parts of $\tilde{f}$ are harmonic in $\mathbf{D}$.). Define $\tilde{f}_r : S^1 \to \mathbf{C}$ by $\tilde{f}_r(z) = \tilde{f}(rz)$. Show that for each $r < 1$, there is a sequence $p_n(z, \bar{z})$ of polynomials in z and $\bar{z}$ such that $p_n(z, \bar{z}) \to \tilde{f}_r(z)$ uniformly for $z \in S^1$. (use Problem 7 of Homework 9.)

   (c) **Weierstrass approximation theorem for S^1.** If $f : S^1 \to \mathbf{C}$ is a continuous function, then there is a sequence $p_n(z, \bar{z})$ of polynomials in z and $\bar{z}$ such that $p_n(z, \bar{z}) \to f(z)$ uniformly for $z \in S^1$.

2. Find a harmonic function on

   (a) the unit disk which has boundary values 0 on the lower semicircle and 1 on the upper semcircle.

   (b) the first quadrant which has boundary values 0 on $[0, 1]$ and 1 on $[1, \infty]$ and $[0, i\infty]$.

3. Use Fourier coefficients to solve the Dirichlet problem in the unit disk for the function on $[0, 2\pi]$: $f(\theta) = -1$ if $\pi/2 < \theta < 3\pi/2$ and 1 otherwise.

4. Suppose that f is an entire function which sends the real line to the real line and the imaginary line to the imaginary line. Prove that f is an odd function, i.e. $f(z) = -f(-z)$. (Hint: We showed that if f sends real line to real line, then $f(z) = \overline{f(\bar{z})}$. Use a similar argument to show that if f sends the imaginary line to the imaginary line the f sends points symmetric with respect to the imaginary axis to points symmetric with respect to imaginary axis.)

5. Suppose that $f(z)$ is holomorphic on $|z| \leq 1$ and satisfies $|f(z)| = 1$ if $|z| = 1$. Show that $f(z)$ is a rational function.