Complex Analysis, Fall 2017

Problem Set 3

Due: September 26 in class

1. Find the linear fractional transformation which maps 1, -1, 0 to 0, i, $-i$.

2. Show that the union of two open connected subsets of $\mathbf{C}$ is open and connected if and only if their intersection is non-empty.

3. Let $H = \{z, \mathrm{Im(z)} > 0\}$ be the upper half plane. Assume $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbf{R})$ is such that $ad - bc > 0$, and let f_M be the corresponding linear fractional transformation. Show that f_M maps H onto H.

4. Assume $(z_1, z_2, z_3, z_4) = \lambda$. What are all the values in terms of λ of cross ratios that we get if we consider all the 24 permutations of z_1, z_2, z_3, z_4? Justify your answer.

5. Let $f = u + iv : U \to \mathbf{C}$ be a function such that the partial derivatives of u and v exist and are continuous on U. Assume f preserves the magnitude of angles at $z_0 \in U$. Show that either f is holomorphic at z_0 with $f'(z_0) \neq 0$, or $\bar{f}$ is holomorphic at z_0 and $\bar{f}'(z_0) \neq 0$.

6. Let $\mathbf{D}$ be the unit disk: $\mathbf{D} = \{z \mid |z| < 1\}$.

   (a) Show that a linear fractional transformation of the of form

   $$f(z) = e^{-i\theta}\,\frac{z - \alpha}{-\bar{\alpha}z + 1}, \quad \alpha \in \mathbf{D},\ \theta \in \mathbf{R}$$

   sends $\mathbf{D}$ to $\mathbf{D}$.

   (b) Conversely show that a linear transformation which sends $\mathbf{D}$ to $\mathbf{D}$ is of the above form. (Hint: Assume α and β are such that $f(0) = \beta$ and $f(\alpha) = 0$. Find $f(\infty)$ and $f^{-1}(\infty)$.)