Complex Analysis, Fall 2017

Problem Set 5

Due: October 12 in class

1. Compute the following integrals: (a) $\int_{|z|=1} \frac{\cos(z^2)}{z}\ dz$ (b) $\int_{|z|=2} \frac{1}{z^2+1}\ dz$ (c) $\int_{|z|=1} e^z z^{-n}\ dz$. (You can only use the facts that we have proved in class.)

2. Let z_0 be a complex number and γ the vertical line segment parametrized by $z(t) = z_0 + it$, $-1 \leq t \leq 1$. Let $\alpha = z_0 + x$ and $\alpha' = z_0 - x$ for positive real number x. Find

$$\lim_{x \to 0^+} \int_\gamma \left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha'}\right) dz.$$

3. Prove the following generalization of Liouville's Theorem: an entire function f which satisfies $|f(z)| < M|z|^n$ for every z with $|z| \geq R_0$ is a polynomial of degree at most n.

4. Assume $f(z)$ is holomorphic on $|z| < 1$ and $|f(z)| \leq \frac{1}{1-|z|}$ on the unit disk. Find the best estimate for $|f^{(n)}(0)|$ that we can get from Cauchy's theorem.

5. Let C be the upper unit semicircle oriented counter clockwise. Compute $\int_\gamma \sqrt{z}\, dz$ in two different ways: using the definition, and using a primitive function.

6. Show that if f is a non-constant entire function, then $f(\mathbf{C})$ is dense in $\mathbf{C}$.

7. A subset $S \subset \mathbf{C}$ is called *star-shaped* if there is a point $z_0 \in S$ such that for every $z \in S$, the line segment joining z_0 and z is in S. Show that a star-shaped region is simply connected.

8. Let U be a bounded region and $\{f_n\}$ a sequence of functions which are holomorphic on U and continuous on the closure of U. Assume that $\{f_n\}$ converges uniformly on the boundary of U. Prove that $\{f_n\}$ converges uniformly on U.