Complex Analysis, Fall 2017

Problem Set 7

Due: November 9 in class

1. Show that every automorphism f of the upper half plane $\mathbf{H}$ is a linear fractional transformation of the form $f = f_M$, where $M \in \mathrm{SL}_2(\mathbf{R})$. We proved this in class for the case $f(i) = i$. If f is arbitrary prove the statement by showing that there is $N \in \mathrm{SL}_2(\mathbf{R})$ such that if $g = f_N \circ f \in \mathrm{Aut}(\mathbf{H})$ and $g(i) = i$. (Assume $f(i) = z$ and show that it is possible to compose a translation by a real number b and a multiplication by a positive real number r to send z to i.)

2. Find the Laurent series of $f(z) = \frac{1}{(z-1)^2(z+1)^2}$ for $1 < |z| < 2$.

3. Let f be holomorphic in the plane except for isolated singularities at $z_1, \ldots, z_n$. We define the residue of f at infinity to be

$$\mathrm{Res}_{z=\infty}(f) = -\frac{1}{2\pi i}\int_C f(z)\,dz$$

where C is a circle of radius R positively oriented around the origin for large R.

(a) Show that $\mathrm{Res}_{z=\infty}(f) = -\sum_{j=1}^{n} \mathrm{Res}_{z=z_j}(f)$

(b) Show that $\mathrm{Res}_{z=\infty}(f)$ is equal to $\mathrm{Res}_{z=0}(-\frac{1}{z^2} f(\frac{1}{z}))$.

4. Prove the following generalization of Schwarz lemma: If $\mathbf{D} = \{z : |z| < 1\}$, and $f : \mathbf{D} \to \mathbf{D}$ is a holomorphic map, then

$$\left|\frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}\right| \leq |z|.$$

5. Suppose that f is a injective holomorphic function from $\mathbf{D}$ onto a square with center 0 with $f(0) = 0$, and assume f is conformal (so $f'(z) \neq 0$ for every $z \in \mathbf{D}$.). Use Schwarz lemma to show that $f(iz) = if(z)$ for all $z \in \mathbf{D}$. (Hint: let $g(z) = f^{-1}(if(z))$.)