Complex Analysis, Fall 2017

Problem Set 8

Due: November 17 in class

1. Find the following residues.

   (a) $\mathrm{Res}_{z=1}\left(\frac{(z^3-1)(z+2)}{(z^4-1)^2}\right)$

   (b) $\mathrm{Res}_{z=0}\left(\frac{\sin z}{z^6}\right)$

   (c) $\mathrm{Res}_{z=1}\left(\frac{1}{z^n-1}\right)$

2. Show that for $n \geq 1$

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}\, dx = \frac{1 \times 3 \times \cdots \times (2n-1)}{2 \times 4 \times \cdots \times (2n)}\, \pi.$$

3. Compute the following integrals over $C = \{|z| = 8\}$ positively oriented.

   (a) $\int_C \frac{1}{1-\cos z}\, dz$

   (b) $\int_C \frac{1+z}{1-e^z}\, dz$

4. Using residues, compute

$$\int_0^{\infty} \frac{\log x}{(x^2+1)^2}\, dx.$$

(Hint: look at $f(z) = \frac{\log z}{(z^2+1)^2}$ where the branch of logarithm is given by deleting the negative imaginary axis, so the angle is $-\pi/2 < \theta < 3\pi/2$. Use the closed path which is the boundary of the upper semicircle of radius R with a bump of radius r avoiding the origin.)

5. Evaluate the integral

$$\int_0^{\pi} \frac{1}{(a+\cos\theta)^2}\, d\theta \qquad (a > 1).$$

6. Suppose that a is a real number which is not an integer. Evaluating the integral $\int_{C_R} \frac{\pi \cos \pi z}{(a+z)^2 \sin \pi z}\, dz$ over the circle C_R of radius $R = N + \frac{1}{2}$, $N \in \mathbf{Z}$, around the origin when $N \to \infty$, show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a+n)^2} = \frac{\pi^2}{(\sin \pi a)^2}.$$

Conclude that

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$