Complex Analysis, Fall 2017

Problem Set 9

Due: November 28 in class

1. (a) Show that if f has a simple pole at z_0, then $\mathrm{Res}_{z=z_0}(f) = \lim_{z\to z_0}(z - z_0)f(z)$ (the limit can be computed by l'Hopital's rule.)

(b) Let $R = \{x + iy \mid -1 \le x \le 1 - \epsilon, 0 \le y \le 1\} \subset \mathbf{C}$ where ϵ is a small positive number, and let γ be the boundary of R. Compute

$$\int_\gamma \frac{1}{z^5 - 1}\, dz.$$

2. Determine the number of zero of the polynomial

$$2z^5 - 6z^2 + z + 1$$

in the annulus $1 \le |z| \le 2$.

3. Let f be holomorphic on the closed unit disk $\overline{\mathbf{D}}$. Assume that $|f(z)| = 1$ if $|z| = 1$, and f is not constant. Use Rouche's theorem to show that

(a) f has a zero in $\mathbf{D}$.

(b) The image of f contains $\mathbf{D}$.

4. Let $f = u + iv$. Show that u and v are both harmonic if and only if $\frac{\partial f}{\partial z}$ is holomorphic.

5. Prove that a harmonic function is an open map.

6. Let $\alpha \in \mathbf{D}$, and $\alpha = re^{i\phi}$.

(a) Show that
$$\text{Re}\,(\frac{e^{i\theta}+\alpha}{e^{i\theta}-\alpha}) = \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}.$$

(b) Show that if u is harmonic on $\mathbf{D}$ and continuous on $\overline{\mathbf{D}}$, then
$$u(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta-\phi)u(e^{i\theta})\,d\theta.$$

where P_r is the *Poisson kernel* given by
$$P_r(\eta) = \frac{1-r^2}{1-2r\cos\eta+r^2}.$$

7. Show that if P_r is as in Problem 7, then for $0 \leq r < 1$ and $\theta \in \mathbf{R}$,
$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|}e^{in\theta}.$$