Math 5021, Fall 2017

Solutions to Midterm exam

1. Show that the function $f = u + iv$,

$$u(x,y) = x^2 - y^2 - 2xy, \quad v(x,y) = x^2 - y^2 + 2xy$$

is everywhere holomorphic. Compute $f'(i)$.

Solution: $u_x = 2x - 2y = v_y$ and $u_y = -2y - 2x = -v_x$, so f is everywhere holomorphic. We have $f' = u_x + iv_x$, so $f'(i) = -2 + 2i$.

2. Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n^2 3^n}{(2n)!} z^{2n}.$$

Solution: The radius of convergence is ∞ since if we let $a_n = \frac{n^2 3^n}{(2n)!}$, then

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \lim_{n\to\infty} \frac{n^2(2n+2)(2n+1)}{3(n+1)^2} = \infty$$

3. (a) Find the linear fractional transformation that sends ∞ to 1 and fixes i and $-i$.
(b) What is the image of the half plane $\mathrm{Re}\, z > 0$ under the map in part (a)?

Solution: (a). If $f = \frac{az+b}{cz+d}$, then $f(\infty) = 1$ implies $a = c$. We can divide by a, b, c, d by a, and assume $a = c = 1$. Then $f(i) = i$, so $i + b = -1 + di$, and $f(-i) = -i$ so $-i + b = -1 - di$. Solving these, we get $b = -1$ and $d = 1$.

(b) The imaginary line $\mathrm{Re}\, z = 0$ is mapped to a line or a circle. We know the image of three points on this line: $f(\infty) = 1$, $f(i) = i$, $f(-i) = -i$. So The image of the imaginary line is the unit circle around the origin. Since $f(1) = 0$, the half plane $\mathrm{Re}\, z > 0$ is mapped to the inside of the unit circle, the unit disk.

4. Compute $\int_C \sqrt{z-1}\ dz$ where C is the unit half circle that joints $1+i$ to $1-i$ clockwise. The principal branch of the square root is used here, so for a complex number w, $\sqrt{w} = e^{\frac{1}{2}\text{Log }w}$.

Solution: $-C$ is parametrized by $z(t) : [-\pi/2, \pi/2]$, $z(t) = 1 + e^{it}$, so

$$\int_C \sqrt{z-1}\ dz = -\int_{-\pi/2}^{\pi/2} ie^{it}e^{\frac{it}{2}}\ dt = -i\frac{2}{3i}e^{\frac{3it}{2}}|_{-\pi/2}^{\pi/2} = -\frac{2}{3}(e^{3\pi i/4} - e^{-3\pi i/4}) = -\frac{2}{3}\sqrt{2}.$$

5. Suppose that f is an entire function such that Re $f(z) \le \frac{2}{|z|}$ for all z such that $|z| \ge 1$. Prove that f is a constant function.

Solution 1: Since Re $f(z) \le \frac{2}{|z|} \le 2$ for $|z| \ge 1$, and since f maps the compact set $\{|z| \le 1\}$ to a compact set, the image of f is the union of a compact set and Re $f(z) \le 2$ which is not dence in $\mathbf{C}$, so by one of the homework problems (HW5) f has to be constant.

Solution 2: Let $g = e^f$. Then g is an entire function and for $|z| \ge 1$, $|e^{g(z)}| = e^{\text{Re } f(z)} \le e^{\frac{2}{|z|}} \le e^2$. Also, over $\{|z| \le 1\}$, $|g(z)|$ obtains its maximum value on the boundary, so for every $z \in \mathbf{C}$, $|e^{g(z)}| \le e^2$. So by Liouville's Theorem g is constant, and hence f is constant.

6. Suppose that $f(z)$ is holomorphic on the disk $|z| < 2$, and $|f(z)| < 1$ for every z such that $|z| < 1$. Show that if $|\omega_1|, |\omega_2| < \frac{1}{2}$, then

$$|f(\omega_1) - f(\omega_2)| \le 4\ |\omega_1 - \omega_2|.$$

Solution: Using the Cauchy's integral formula for the unit circle C positively oriented, we get

$$|f(\omega_1) - f(\omega_2)| = |\frac{1}{2\pi i}\int_C \frac{f(z)}{z-\omega_1} - \frac{f(z)}{z-\omega_2}\ dz| = |\frac{1}{2\pi i}\int_C \frac{\omega_1-\omega_2}{(z-\omega_1)(z-\omega_2)}\ dz|$$
$$\le \frac{|\omega_1-\omega_2|}{2\pi}\ 2\pi \max_{z\in C}\frac{1}{|(z-\omega_1)(z-\omega_2))|} = 4|\omega_1-\omega_2|.$$

7. (a) Define a removable singularity, a pole, and an essential singularity.

(b) Suppose that f has an isolated singularity at z_0, and $f(z) \notin (-\infty, 0]$ for any z near z_0. Show that z_0 is a removable singularity of f.

Solution: (b) Since the image of f near z_0 does not intersect $(-\infty, 0]$, it is possible to define a single valued holomorphic function $g = f^{1/m}$ for every $m \geq 2$ in such a way that the image of g near z_0 is in $\{z : -\pi/m < \arg z < \pi/m\}$. If z_0 is a pole (essential singularity) of f, then z_0 would be a pole (essential singularity) of g by definition. This immediately shows z_0 cannot be an essential singularity of f since the image of g is not dense in $\mathbf{C}$. Also since $f = g^m$, if z_0 is a pole of order n for g, it would be a pole of order mn for f. So if we assume z_0 is a pole of order k for f, and we pick $m > k$, we get a contradiction.