Analysis I, Fall 2017

Solutions to Problem Set 1

5. Write $f \circ g = u + iv$, $g = u_1 + iv_1$, and $f = u_2 + iv_2$. Then

$$\frac{\partial(f \circ g)}{\partial z} = \frac{1}{2}[(u_x + v_y) + i(v_x - u_y)].$$

The chain rule for partial derivatives (thinking of f and g as maps from $\mathbf{R}^2$ to $\mathbf{R}^2$) gives

$$\begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_{2x} & u_{2y} \\ v_{2x} & v_{2y} \end{bmatrix} \begin{bmatrix} u_{1x} & u_{1y} \\ v_{1x} & v_{1y} \end{bmatrix}$$

So

$$\begin{aligned} \frac{\partial(f \circ g)}{\partial z} &= \frac{1}{2}[(u_{2x}u_{1x} + u_{2y}v_{1x} + v_{2x}u_{1y} + v_{2y}v_{1y}) + i(v_{2x}u_{1x} + v_{2y}v_{1x} - u_{2x}u_{1y} - u_{2y}v_{1y})] \\ &= \frac{1}{4}[(u_{2x} + v_{2y} + i(v_{2x} - u_{2y}))(u_{1x} + v_{1y} + i(v_{1x} - u_{1y})) + \\ &\quad (u_{2x} - v_{2y} + i(v_{2x} + u_{2y}))(u_{1x} - v_{1y} + i(-v_{1x} - u_{1y}))] \\ &= \frac{\partial f}{\partial z}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial \bar{z}}\frac{\partial \bar{g}}{\partial z} \end{aligned}$$

6. (a) Write $f = u + iv$. Since u is constant on U, $u_x = u_y = 0$. So by Cauchy-Riemann equations $v_x = v_y = 0$, so $f' = 0$, so f is a constant.

(b) Assume $|f| = c$ on U. If $c = 0$, then $f = 0$ on U. Assume $c \neq 0$. Then $f\bar{f} = c^2$. Taking $\frac{\partial}{\partial z}$ we get

$$\frac{\partial f}{\partial z}\bar{f} + f\frac{\partial \bar{f}}{\partial z} = 0.$$

But $\frac{\partial \bar{f}}{\partial z} = \overline{\frac{\partial f}{\partial \bar{z}}}$. Since f is holomorphic, $\frac{\partial f}{\partial \bar{z}} = 0$, so $\frac{\partial \bar{f}}{\partial z} = 0$. Therefore, $\frac{\partial f}{\partial z}\bar{f} = 0$. Since we are assuming $c \neq 0$, $\bar{f} \neq 0$, so $\frac{\partial f}{\partial z} = 0$ on U. Again since f is holomorphic $f' = \frac{\partial f}{\partial z} = 0$ on U, so f is constant on U.

7. This follows from the proof of Lucas' Theorem. Let $z_1, \ldots, z_n$ be the roots of f and w a root of f', so

$$f = a_n(z - z_1) \ldots (z - z_n).$$

If $f(w) = 0$, then w is a repeated root of f, and we are done. Otherwise, assume $f(w) \neq 0$, and write

$$0 = \frac{f'(w)}{f(w)} = \sum_{i=1}^{n} a_n \frac{1}{w - z_i}. \tag{1}$$

Let L be the line formed by the boundary of $\Delta(f)$ which contains w. Then there is half-plane formed by L which contains all the roots of f. Assume the equation of that half-plane is

$$\{z : Im((w - z)e^{-i\theta}) \geq 0\}.$$

Then equation of the line is $\{z : Im((z - w)e^{-i\theta}) = 0\}$. If all the roots of f lie on L, then we are done. Otherwise, there is z_i such that $Im((w - z_i)e^{-i\theta}) > 0$, so $Im(\frac{e^{i\theta}}{w - z_j}) < 0$. Multiplying both sides of Equation 1 by $e^{i\theta}$ we get a contradiction since the imaginary part of each term would be non-positive, and the imaginary part of at least one term is strictly negative.

8. First Assume B is a finite number. Then for every $\epsilon > 0$, there is N such that for $n \geq N$, $B - \epsilon \leq |\frac{a_n}{a_{n+1}}| \leq B + \epsilon$. Therefore for every $k \geq N$

$$(B - \epsilon)^{k-N} \leq \ |\frac{a_N}{a_k}| \ \leq (B + \epsilon)^{k-N}.$$

So

$$\frac{|a_N|}{(B + \epsilon)^{k-N}} \leq \ |a_k| \ \leq \frac{|a_N|}{(B - \epsilon)^{k-N}}.$$

Therefore for every $k \geq N$

$$\frac{|a_N|^{\frac{1}{k}}}{(B + \epsilon)^{1 - \frac{N}{k}}} \leq \ |a_k|^{\frac{1}{k}} \ \leq \frac{|a_N|^{\frac{1}{k}}}{(B - \epsilon)^{1 - \frac{N}{k}}}.$$

So

$$\frac{1}{B - \epsilon} \leq \ \limsup_{k \to \infty} |a_k|^{\frac{1}{k}} \ \leq \frac{1}{B + \epsilon}.$$

Letting ϵ go to zero, we get $\limsup_{k \to \infty} |a_k|^{\frac{1}{k}} = \frac{1}{B}$.

Similarly if $B = \infty$, then for every $M > 0$, there is N such that for $n \geq N$, $|\frac{a_n}{a_{n+1}}| \geq M$. So for every $k \geq N$,

$$|\frac{a_N}{a_k}| \geq M^{k-N}.$$

So for every $k \geq N$,

$$|a_k|^{\frac{1}{k}} \leq \frac{|a_N|^{\frac{1}{k}}}{M^{1-\frac{1}{k}}}.$$

Therefore,

$$\limsup_{k\to\infty} |a_k|^{\frac{1}{k}} \leq \frac{1}{M},$$

so $\limsup_{k\to\infty} |a_k|^{\frac{1}{k}} = 0$.