Complex Analysis, Fall 2017

Solutions to Problem Set 10

1. (a) We need to show that for every $\epsilon > 0$, there is δ such that for every r with $0 \leq 1 - r < \delta$ and every $z \in S^1$, $|g(z) - g_r(z)| < \epsilon$. Since g is continuous on $\overline{\mathbf{D}}$, and $\overline{\mathbf{D}}$ is compact, f is uniformly continuous on $\overline{\mathbf{D}}$. So for every $\epsilon > 0$, there is δ, such that if $|z - z'| < \delta$ and $z, z' \in \overline{\mathbf{D}}$, then $|g(z) - g(z')| < \epsilon$. So if $0 \leq 1 - r < \delta$, and $z = e^{i\theta} \in S^1$, then $|z - rz| = 1 - r < \delta$, and so $|g(z) - g_r(z)| = |g(z) - g(rz)| < \epsilon$.

(b) By Hw 9, Question 7,

$$P_r(\theta - \phi) = 1 + \sum_{n=1}^{\infty} r^n (e^{in(\theta-\phi)} + e^{-in(\theta-\phi)})$$

and it follows from the solution that for a fixed r this series is uniformly convergent for $\theta - \phi \in \mathbf{R}$ (since the geometric series is uniformly convergent on a circle of radius $r < 1$.) Therefore if we let $a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, d\theta$, then

$$\begin{aligned}\tilde{f}_r(z) = \tilde{f}(re^{i\phi}) &= a_0 + \sum_{n=1}^{\infty} \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, r^n (e^{in(\theta-\phi)} + e^{-in(\theta-\phi)})\, d\theta \\ &= a_0 + \sum_{n=1}^{\infty} (re^{i\phi})^n (\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, e^{-in\theta}\, d\theta) + (re^{-i\phi})^n (\frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, e^{in\theta}\, d\theta).\end{aligned}$$

So if we let $a_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta}\, d\theta$ for $n \geq 0$ and $b_n = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{in\theta}\, d\theta$ for $n \geq 1$, then

$$\tilde{f}_r(z) = a_0 + \sum_{n=1}^{\infty} (a_n \bar{z}^n + b_n z^n)$$

and therefore if $p_n(z, \bar{z}) = 1 + \sum_{j=1}^{n} a_j \bar{z}^j + \sum_{j=1}^{\infty} b_j z^j$, we see that $p_n(z, \bar{z}) \to \tilde{f}_r(z)$ uniformly for $z \in S^1$. Note that p_n is independent of r.

(c) It is easy to see now that $p_n(z, \bar{z}) \to f(z)$ uniformly: if $\epsilon > 0$ is given, there is δ such that for $0 < 1 - r < \delta$, and $z \in S^1$, $|f(z) - \tilde{f}_r(z)| < \epsilon/2$. Pick such a r. Then there is N such that for $n \geq N$, $|p_n(z, \bar{z}) - \tilde{f}_r(z)| < \epsilon/2$ for every $z \in S^1$. So $|f(z) - p_n(z, \bar{z})| < \epsilon$ for all $z \in S^1$.

2. (a) Let f be a linear fractional transformation which sends the unit disk to the upper half plane (and therefore the unit circle to the real line). One such linear function is $f(z) = \frac{z-1}{iz+i}$. This functions sends the upper half circle to the positive side of the real line and the lower half circle to the negative side. The function $u = \frac{\arg z}{\pi}$ $-\pi/2 \leq \arg z < 3\pi/2$ is harmonic on $\mathbf{C} \setminus \{i\mathbf{R}_{\leq 0}\}$. It is piece-wise continuous on the real line, and has value 0 on the positive part and 1 on the negative side. Since the composition of a holomorphic function with a harmonic function is harmonic, $u \circ f$ is harmonic on the unit disk, and takes value 0 on the upper half circle and 1 on the lower one. Therefore $1 - u \circ f = 1 - \frac{1}{\pi}\arg(\frac{z-1}{iz+i})$ is the desired harmonic function.

(b) This is done similar to part (a). First notice that the function $f(z) = z^2$ sends the first quadrant to the upper half plane, the interval $(0,1)$ to $(0,1)$, the positive part of the imaginary axis to the negative part of the real axis, and $(1,\infty)$ to $(1,\infty)$. Now it is enough to find a linear fractional transformation g which sends 0 to 0, 1 to ∞, and ∞ to -1. Then g would send the upper half plan to the upper half plane (since it preserves orientation), $(0,1)$ to $(0,\infty)$, $(1,\infty)$ to $\{x \leq -1\}$, and the negative part of the real line to $(-1,0)$. One such linear fractional transformation is $g(z) = \frac{z}{-z+1}$. Now $g \circ f$ is a conformal map which sends the the first quadrant to the upper half plane, $(0,1)$ to $(0,\infty)$, $(1,\infty) \cup i\mathbf{R}_{\geq 0}$ to the negative part of the real axis. The function $\frac{1}{\pi}\arg(z)$ as in part (a) is harmonic on the upper half plane with value 0 and 1 on the positive and negative part of the real axis respectively, so the function $u \circ g \circ f = \frac{1}{\pi}\arg(\frac{z^2}{-z^2+1})$ is the harmonic function on the first quadrant with the desired values on the boundary.

4. We show that if f sends the imaginary axis to the imaginary axis, then f sends points symmetric with respect to the imaginary axis to points symmetric with respect to imaginary axis. For any $z \in \mathbf{C}$, z and $-\bar{z}$ are symmetric with respect to the imaginary axis. So it is enough to show that for every z, $f(-z) + \overline{f(\bar{z})} = 0$. Clearly, $f(-z) + \overline{f(\bar{z})}$ is holomorphic which is zero on the imaginary axis by our assumption, and since the zeros of a holomorphic function are not isolated only when the function is identical to zero, we get the desired result.

Also, we have shown that since f sends the real line to the real line, $f(z) = \overline{f(\bar{z})}$, so

$$f(-z) = -\overline{f(\bar{z})} = -f(z).$$

5. By Schwarz reflection principle, we can extend f to a meromorphic function on $\mathbf{C}$. If $f(0) \neq 0$, then ∞ is a removable singularity for f, and if $f(0) = 0$, then ∞ is a pole for f, so f is meromorphic on $\hat{\mathbf{C}}$. Hence by a result earlier proved in class, f is a rational function.