Analysis I, Fall 2017

Solutions to Problem Set 3

3. Write $z = x + iy$, then

$$f_M(z) = \frac{(az+b)(c\bar{z}+d)}{|cz+d|^2} = \frac{(ax+b+iya)(cx+d-iyc)}{|cz+d|^2}.$$

So

$$|cz+d|^2 \, \mathrm{Im} f_M(z) = ya(cx+d) - yc(ax+b) = y(ad-bc).$$

Hence the imaginary part of $f_M(z)$ is positive if $y > 0$.

4. Let $CR(z_1, z_2, z_3, z_4) = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} = \lambda$. Then we have

$$CR(z_2, z_1, z_4, z_3) = CR(z_4, z_3, z_2, z_1) = CR(z_3, z_4, z_1, z_2) = \lambda.$$

So we we only need to consider 6 different permutations: Every other permutation has a cross ratio equal to one of the following.

- $CR(z_1, z_2, z_3, z_4) = \lambda$
- $CR(z_1, z_2, z_4, z_3) = \frac{(z_1-z_4)(z_2-z_3)}{(z_1-z_3)(z_2-z_4)} = \frac{1}{\lambda}$
- $CR(z_1, z_3, z_2, z_4) = 1 - \lambda$
- $CR(z_1, z_3, z_4, z_2) = \frac{1}{1-\lambda}$
- $CR(z_1, z_4, z_3, z_2) = 1 - \frac{1}{1-\lambda} = \frac{\lambda}{\lambda-1}$
- $CR(z_1, z_4, z_2, z_3) = \frac{\lambda-1}{\lambda}$

5. Let $f = u + iv$, and $A = \begin{bmatrix} u_x(x_0, y_0) & u_y(x_0, y_0) \\ v_x(x_0, y_0) & v_y(x_0, y_0) \end{bmatrix}$. If C is a smooth curve given by parametrization $\gamma(t) = x(t) + iy(t) : [a, b] \to \mathbf{C}$, and $\gamma(t_0) = z_0$, then the tangent vector to C at z_0 is given by $v = \begin{bmatrix} x'(t_0) \\ y'(t_0) \end{bmatrix}$, and by the chain rule the tangent vector

to $f \circ \gamma$ is given by Av. So A is a 2 by 2 real matrix which preserves angles. Then suppose that $A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ has angle θ with the positive direction of the real axis. If

$$B_{-\theta} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is the matrix of rotation by $-\theta$, then $C = B_{-\theta}A$ also preserves angles and sends $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to a vector $\begin{bmatrix} r \\ 0 \end{bmatrix}$ with $r > 0$. Therefore it sends the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 0 \\ s \end{bmatrix}$ for some non-zero $s \in \mathbf{R}$. So

$$C = \begin{bmatrix} r & 0 \\ 0 & s \end{bmatrix}.$$

Then $C \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ should be of the form $\begin{bmatrix} \lambda \\ \lambda \end{bmatrix}$ or $\begin{bmatrix} \lambda \\ -\lambda \end{bmatrix}$, so $r = s$ or $r = -s$ and hence

$$A = \begin{bmatrix} r\cos\theta & r\sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ -r\sin\theta & -r\cos\theta \end{bmatrix}$$

So either $u_x = v_y, v_x = -u_y$ or $u_x = -v_y, v_x = u_y$. So either $u + iv$ satisfies the Cauchy-Reimann equations or $u - iv$ does, and since the partial derivatives are continuous, this implies either f or $\bar{f}$ is holomorphic at z_0. It is clear that in either case the matrix A is non-zero.

6. (a) It is enough to show if $|z| < 1$, then $|z - \alpha|^2 < |-\bar{\alpha}z + 1|^2$. We have

$$\begin{aligned} |-\bar{\alpha}z + 1|^2 - |z - \alpha|^2 &= (-\bar{\alpha}z + 1)(-\alpha\bar{z} + 1) - (z - \alpha)(\bar{z} - \bar{\alpha}) \\ &= \alpha\bar{\alpha}z\bar{z} - \alpha\bar{\alpha} - z\bar{z} + 1 \\ &= (\alpha\bar{\alpha} - 1)(z\bar{z} - 1) \\ &= (|\alpha|^2 - 1)(|z|^2 - 1) > 0 \end{aligned}$$

since α and z both belong to the unit circle.

(b) Let $f(z) = \frac{az+b}{cz+d}$. Let α be so that $f(\alpha) = 0$, then since α and $\frac{\alpha}{|\alpha|^2}$ are conjugates with respect to the unit circle, $f(\frac{\alpha}{|\alpha|^2})$ should be the conjugate of zero which is infinity. Also if $f(0) = \beta$, then $f(\infty) = \frac{\beta}{|\beta|^2}$. So we have the following relations:

$$\alpha = -\frac{b}{a}, \quad \beta = \frac{b}{d}, \quad \frac{\alpha}{|\alpha|^2} = -\frac{d}{c}, \quad \frac{\beta}{|\beta|^2} = \frac{a}{c}.$$

Multiplying the first and last identities, and then multiplying the middle two identities, we get $\frac{\alpha\beta}{|\beta|^2} = -\frac{b}{c} = \frac{\beta\alpha}{|\alpha|^2}$., so $|\alpha| = |\beta|$. Let $\beta = e^{i\phi}\alpha$. Then $b = -\alpha a$, $c = a|\beta|^2/\beta = e^{-i\phi}a|\alpha|^2/\alpha = e^{-i\phi}a\bar{\alpha}$, and $d = -c\alpha/|\alpha|^2 = -c/\bar{\alpha} = -e^{-i\phi}a$. If we plug in these identities, we get

$$f(z) = -e^{i\phi}\frac{z-\alpha}{-\bar{\alpha}z+1}.$$

Set $\theta = -(\pi+\phi)$. Then $e^{-i\theta} = -e^{i\phi}$, and we get the desired form.