Complex Analysis I, Fall 2017

Solutions to Problem Set 4

1. Since 0 and ∞ are conjugate with respect to the circle $|z| = 2$, the linear fractional transformation should send ∞ to the conjugate of i with respect to the circle $|z+1| = 1$ which is $w = \frac{\sqrt{2}}{2} e^{\frac{3\pi i}{4}}$.

So

$$f = \frac{wz + 2w}{z - 2iw} = \frac{z + 2}{\frac{z}{w} - 2i}.$$

3. We have

$$\sqrt{1+z} + \sqrt{1-z} = e^{\frac{1}{2}\log(1+z)} + e^{\frac{1}{2}\log(1-z)}.$$

Since the exponential function is holomorphic everywhere on $\mathbf{C}$, the function $\sqrt{1+z} + \sqrt{1-z}$ is holomorphic whenever $\log(1+z)$ and $\log(1-z)$ are both holomorphic. For example, since $\log(1+z)$ is holomorphic on $\mathbf{C} \setminus \mathbf{R}_{\leq -1}$ and $\log(1-z)$ is holomorphic on $\mathbf{C} \setminus \mathbf{R}_{\geq 1}$, $\sqrt{1+z} + \sqrt{1-z}$ is holomorphic on $\mathbf{C} \setminus (\mathbf{R}_{\leq -1} \cup \mathbf{R}_{\geq 1})$ which is an open set.

4. Set $w = \tan z$. Then

$$w = \tan z = \frac{\sin z}{\cos z} = -i \, \frac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}} = -i \, \frac{e^{2iz} - 1}{e^{2iz} + 1}.$$

Set $\alpha = e^{iz}$. Then $w = -i \, \frac{\alpha^2 - 1}{\alpha^2 + 1}$, so $(\alpha^2 + 1)w = -i(\alpha^2 - 1)$, therefore, $\alpha^2(w + i) = (i - w)$. So

$$e^{2iz} = \frac{i - w}{i + w}.$$

and $2iz = \log \frac{i-w}{i+w}$. So

$$\arctan w = -\frac{i}{2} \log(\frac{i - w}{i + w}).$$

5. (a) Let C be a circle passing through 1. Then $f(C)$ is either a circle or a line in $\mathbf{C}$. Since $f(1) = \infty$, the image of $f(C)$ is a line. Since C and the real axis are orthogonal

by our assumption, and since linear fractional transformations preserve angles, $f(C)$ should be orthogonal to the imaginary axis.

(b) First we construct a linear fractional transformation f such that $f(1) = \infty, f(0) = i, f(-1) = 0$. Such a function will send the circle $|z| = 1$ to the real line and the circle $|z - \frac{1}{2}| = \frac{1}{2}$ to the line $y - i$ by part (a). Solving the system of linear equations we find

$$f(z) = \frac{z+1}{iz - i}.$$

Since $f(0) = i$, f sends inside of circle $|z| = 1$ to the upper half plane, and since $f(-1) = 0$ f sends the half plane $y < 1$ to the outside of the circle $|z - \frac{1}{2}| = \frac{1}{2}$. Therefore the area between the two circles is mapped to the strip $0 < y < 1$. Any linear fractional transformation $f(z) = \frac{az+b}{cz+d}$ is bijective and it is conformal at $z \neq -d/c$.

6. (a) If $z = x+iy$, then $f(z) = \frac{1}{2}(x+\frac{x}{x^2+y^2}+i(y-\frac{y}{x^2+y^2}))$. Therefore is $f(z) \in [-1,1]$, then Im f(z) = 0, so $x^2 + y^2 = 0$ or $y = 0$. If $y = 0$, then $-2 \leq f(z) = x + \frac{1}{x} \leq 2$, so if z is outside the unit circle, $f(z)$ does not belong to $[-1,1]$, and if z is on the unit circle, then $f(z) \in [-1,1]$.

If $w = \frac{1}{2}(z + \frac{1}{z})$, then $z^2 - 2wz + 1 = 0$. Therefore $(w - z)(w - \frac{1}{z}) = 0$. (and $f^{-1}(w) = \{z, \frac{1}{z}\}$.) If $|z| > 1$, then $|\frac{1}{z}| < 1$, so f is a bijection from the outside of the unit circle to $\mathbf{C} \setminus [-1,1]$. And $f'(z) = \frac{z^2-1}{z^2}$ which is not zero outside the unit circle, so f is conformal outside the unit circle.

(b) Note that in part (a), the function f sends the region outside the unit circle in the second quadrant bijectively to the second quadrant (since $x < 0$ and $y < 0$, and $x^2 + y^2 > 1$, $x + \frac{x}{x^2+y^2} < 0$, and $y - \frac{y}{x^2+y^2} > 0$) We have

$$\sin z = -\frac{i}{2}(e^{iz} - e^{-iz}) = -(\frac{1}{2}(ie^{iz} + \frac{1}{ie^{iz}})).$$

Let S_1 be the strip $0 < x < \pi/2, y < 0$. Then

- $z \mapsto iz$ maps S_1 to bijectively onto the strip $S_2 = \{z :\ 0 < y < \pi/2, x > 0\}$.
- $z \mapsto e^{iz}$ maps S_1 bijectively onto the region $S_3 = \{z :\ |z| > 1, x, y > 0\}$.
- $z \mapsto ie^{-z}$ maps S_1 bijectively onto the region $S_4 = \{z :\ |z| > 1, x < 0, y > 0\}$.
- $z \mapsto \frac{1}{2}(ie^{-z} + \frac{1}{ie^{iz}})$ maps the region S_1 bijectively onto the second quadrant by part (a)
- $z \mapsto \sin z$ sends S_1 bijectively onto the fourth quadrant $x > 0, y < 0$.