Complex Analysis, Fall 2017

Solutions to Problem Set 5

1. (a)
$$\int_{|z|=1} \frac{\cos(z^2)}{z}\,dz = 2\pi i\cos(0) = 2\pi i.$$

   (b)
$$\int_{|z|=2} \frac{1}{z^2+1}\,dz = \frac{1}{2i}\int_{|z|=2} \frac{1}{z-i}\,dz - \frac{1}{2i}\int_{|z|=2} \frac{1}{z+i}\,dz = \frac{1}{2i}(2\pi i - 2\pi i) = 0.$$

   (c)
$$\int_{|z|=1} \frac{e^z}{z^n}\,dz = \frac{2\pi i}{(n-1)!}(e^z)^{(n-1)}(0) = \frac{2\pi i}{(n-1)!}.$$

2. We have

$$\begin{aligned}
\lim_{x\to 0^+}\int_\gamma \left(\frac{1}{z-\alpha} - \frac{1}{z-\alpha'}\right) dz &= \lim_{x\to 0^+}\int_{-1}^{1} \left(\frac{1}{z(t)-\alpha} - \frac{1}{z(t)-\alpha'}\right) z'(t)\,dt \\
&= i\lim_{x\to 0^+}\int_{-1}^{1}\left(\frac{1}{it-x} - \frac{1}{it+x}\right) dt. \\
&= -2i\lim_{x\to 0^+} x\int_{-1}^{1} \frac{1}{t^2+x^2}\,dt \\
&= -2i\lim_{x\to 0^+} x\frac{1}{x}\arctan\left(\frac{t}{x}\right)\Big|_{-1}^{1} \\
&= -2i\lim_{x\to 0^+}\left(\arctan\left(\frac{1}{x}\right) - \arctan\left(\frac{-1}{x}\right)\right) \\
&= -2i\left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = -2i\pi.
\end{aligned}$$

3. By the generalized version of Cauchy's integral formula if C is circle of radius r around 0 which is positively oriented and z_0 is such that $|z_0| < r$, then

$$f^{(n+1)}(z_0) = \frac{(n+1)!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+2}}\,dz$$

Therefore,

$$|f^{(n+1)}(z_0)| \leq \frac{(n+1)!}{2\pi}\text{length}(C) \max_{z\in C} \frac{|f(z)|}{r^{n+2}} = \frac{(n+1)!}{r^{n+1}} \max_{z\in C} |f(z)| \leq \frac{M(n+1)!}{r}$$

Letting $r \to \infty$, we see that $f^{(n+1)}(z_0) = 0$ for every z_0. Therefore, f is a polynomial of degree at most n.

4. For any $r < 1$, and let C be the circle of radius r around 0 positively oriented. We have

$$\begin{aligned}
|f^{(n)}(0)| = |\frac{n!}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\, dz| &\leq |\frac{n!}{2\pi i}|\ \text{length}(C) \max_{z\in C} |\frac{f(z)}{z^{n+1}}| \\
&\leq \frac{n!}{2\pi}(2\pi r) \max_{z\in C} \frac{1}{(1-|z|)|z^{n+1}|} \\
&= \frac{n!}{2\pi}\frac{1}{(1-r)r^{n+1}}(2\pi r) \\
&= \frac{n!}{(1-r)r^n}.
\end{aligned}$$

The function $(1-r)r^n$ takes its maximum at $r = \frac{n}{n+1}$ in the interval $[0,1]$, so the minimum value of $\frac{n!}{(1-r)r^n}$ is

$$\frac{(n+1)!(n+1)^n}{n^n}.$$

5. There are two different values of $\sqrt{z}$ that we can consider in this region to get a holomorphic function. One choice would be the root of z in the upper half plane, so $z^{1/2} = e^{\frac{1}{2}\log z}$ where

$$\log z = \log|z| + i\arg z, \quad 0 \leq \arg z \leq \pi.$$

And a primitive would be $\frac{2}{3}z^{3/2} = \frac{2}{3}e^{\frac{3}{2}\log z}$. For this choice we get:

If we use the definition, we get

$$\int_\gamma \sqrt{z}\, dz = i\int_0^\pi \sqrt{e^{it}}e^{it}\, dt = i\int_0^\pi e^{3it/2}\, dt = \frac{2}{3}(e^{3i\pi/2} - 1) = \frac{2}{3}(-i-1).$$

If we use the primitive we get

$$\int_\gamma \sqrt{z}\, dz = \frac{2}{3}\int_\gamma (z^{3/2})'\, dz = \frac{2}{3}(i^{3/2} - 1) = \frac{2}{3}(-i-1).$$

6. Assume to the contrary that $f(\mathbf{C})$ is not dense. Then there is $z_0 \in \mathbf{C}$, and $\epsilon > 0$ such that $f(C) \cap D(z_0, \epsilon) = \emptyset$. Then the function $g(z) = \frac{1}{f(z)-z_0}$ is also entire, and is bounded $|\frac{1}{f(z)-z_0}| < \frac{1}{\epsilon}$. So by Liouville's theorem, $g(z)$ is constant, and therefore $f(z)$ is constant, a contradiction.

7. Let $\gamma : [0,1] \to \mathbf{C}$ be a path with initial point p and end point q. We show that γ is homotopic to the path which goes first from p to z_0 on a straight line and then z_0 to q on a straight line, that is

$$\gamma' : [0,1] \to \mathbf{C}, \quad \gamma'(t) = \begin{cases} (1-2t)p + (2t)z_0 & \text{for } 0 \le t \le \frac{1}{2} \\ (2-2t)z_0 + (2t-1)q & \text{for } \frac{1}{2} \le t \le 1 \end{cases}$$

It is easy to see that being homotopic is an equivalence relation on the set of paths with initial point p and end point q, and therefore this will prove that U is simply connected.

Now if γ and γ' are as above, the following map

$$F : [0,1] \times [0,1] \to \mathbf{C} : F(s,t) = \begin{cases} (1-2t)p + (2t)z_0 & 0 \le t \le \frac{s}{2} \\ (1-s)\gamma(\frac{t-\frac{s}{2}}{1-s}) + sz_0 & \frac{s}{2} \le t \le 1 - \frac{s}{2} \\ (2-2t)z_0 + (2t-1)q & 1 - \frac{s}{2} \le t \le 1. \end{cases}$$

is a continuous function such that $F(0,t) = \gamma(t)$, $F(1,t) = \gamma'(t)$ for all t, and $F(s,0) = p$ and $F(s,1) = q$ for all s.

8. For every ϵ, there is N such that for $n, m \ge N$, $|f_n(z) - f_m(z)| < \epsilon$ for all z in the boundary of U. But since $\bar{U}$ is bounded and closed, it is compact, and since $|f_n(z) - f_m(z)|$ is continuous on $\bar{U}$, it takes its maximum value on $\bar{U}$. Since $f_n - f_m$ is holomorphic, the maximum modulus should be obtained on the boundary, and therefore $|f(z) - f_m(z)| < \epsilon$ for all $z \in U$. Therefore $\{f_n\}$ is uniformly convergent on U.