Complex Analysis, Fall 2017

Solutions to Problem Set 7

1. We have proved the statement for automorphisms f such that $f(i) = i$. Assume now $f(i) = z_0$. It is enough to show that there is a matrix $M \in \mathrm{SL}_2(\mathbf{R})$ such that $f_M(z_0) = i$, since this gives $f_M \circ f(i) = i$, and therefore $f_M \circ f = f_N$ for some $N \in \mathrm{SL}_2(\mathbf{R})$, so $f = f_M^{-1} \circ f_N = f_{M^{-1}N}$.

Now note that if $z_0 = a + ib$, and

$$A = \begin{bmatrix} 1 & -a \\ 0 & 1 \end{bmatrix},$$

then $f_A(z_0) = ib$. And if $r = \frac{1}{\sqrt{b}}$, and

$$B = \begin{bmatrix} r & 0 \\ 0 & \frac{1}{r} \end{bmatrix},$$

then $f_B(ib) = i.$, so $f_{BA}(z_0) = i$ and $B, A \in \mathrm{SL}_2(\mathrm{R})$.

2. For $|z| > 1$, we have

$$\begin{aligned} f(z) = \frac{1}{(z^2-1)^2} = \frac{1}{4z}\left(\frac{1}{(z-1)^2} - \frac{1}{(z+1)^2}\right) &= \frac{1}{4z^3}\left(\frac{1}{(1-1/z)^2} - \frac{1}{(1+1/z)^2}\right) \\ &= \frac{1}{4z^3}\left(\sum_{n\geq 1} \frac{n}{z^{n-1}} - \sum_{n\geq 1} (-1)^n \frac{n}{z^{n-1}}\right) \\ &= \frac{1}{4z^3}\left(\sum_{n\geq 1} (1+(-1)^n)\frac{n}{z^{n-1}}\right) \\ &= \sum_{n\geq 1} \frac{n}{z^{2n+2}}. \end{aligned}$$

3. (a) Let $c_n = \mathrm{Res}_{z=z_j}(f)$, and let R be sufficiently large so that $|z_j| < R$ for all j. Then residue formula gives

$$\int_{|Z|=R} f(z)\, dz = 2\pi i \sum_k c_k$$

so

$$\mathrm{Res}_{z=\infty}(f) + \sum_k \mathrm{Res}_{z=z_k}(f) = 0.$$

(b) Let R be large enough so that all the poles of $f(z)$ are outside the circle $|z| = R$, and so $f(\frac{1}{z})$ is holomorphic on $\{z : 0 < |z| < \frac{1}{R}\}$ Then the residue theorem for $g(z) := \frac{-1}{z^2} f(\frac{1}{z})$ gives:

$$\begin{aligned}
\mathrm{Res}_{z=0}(g) = \int_{|z|=\frac{1}{R}} \frac{-1}{z^2} f(\frac{1}{z})\, dz &= \frac{-1}{2\pi i} \int_0^{2\pi} R^2 e^{2i\theta} f(Re^{-i\theta}) \frac{1}{R} i e^{-i\theta}\, d\theta \\
&= \frac{-1}{2\pi i} \int_0^{2\pi} iRe^{-i\theta} f(Re^{-i\theta})\, d\theta \\
&= \frac{1}{2\pi i} \int_0^{2\pi} iRe^{i\theta} f(Re^{i\theta})\, d\theta \\
&= \mathrm{Res}_{z=\infty}(f).
\end{aligned}$$

4. We let $\alpha = f(0)$ and $g = \psi_\alpha \circ f$, where $\psi_\alpha \in \mathrm{Aut}(\mathbf{D})$

$$\psi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Then we can apply Schwarz lemma to $g(z)$ since $g(0) = 0$.

5. Let $g(z) = f^{-1}(if(z))$. The

$$g'(z) = \frac{if'(z)}{f'(f^{-1}(if(z)))}.$$

(f is conformal everywhere, so f^{-1} is holomorphic and $(f^{-1})'(w) = \frac{1}{f'(f^{-1}(w))}$ by the chain rule.) Since $f(0) = 0$ and f is injective, $f^{-1}(0) = 0$ which implies that $g(0) = 0$. Also, $g'(0) = \frac{if'(0)}{f'(0)} = i$, so since $|g'(0)| = 1$, Schwarz lemma gives $g(z) = g'(0)z = iz$, and so $f(iz) = if(z)$.