Complex Analysis, Fall 2017

Solutions to Problem Set 9

1. (b) Let $R = \{x + iy : -1 \le x \le 1 - \epsilon, 0 \le y \le 1\} \subset \mathbf{C}$ with $\epsilon > 0$ small. The function $f(z) = \frac{1}{z^5-1}$ is meromorphic with poles at $z = e^{2\pi ik/5}$ $k = 0, 1, \ldots, 4$. Since $e^{2\pi ik/5} = \cos(2\pi k/5) + i\sin(2\pi k/5)$, $-1 \le \cos(2\pi k/5) \le 1 - \epsilon$ for $k = 1, 2, 3, 4$ and $0 \le \sin(2\pi k/5) \le 1$ for $k = 0, 1, 2$, then poles of f in R are $e^{2\pi i/5}$ and $e^{4\pi i/5}$. We have

$$\mathrm{Res}_{z=e^{2\pi i/5}}(f) = \lim_{z \to e^{2\pi i/5}} \frac{z - e^{2\pi i/5}}{z^5 - 1} = \frac{1}{5} e^{2\pi i/5}.$$

and

$$\mathrm{Res}_{z=e^{4\pi i/5}}(f) = \lim_{z \to e^{4\pi i/5}} \frac{z - e^{4\pi i/5}}{z^5 - 1} = \frac{1}{5} e^{4\pi i/5}.$$

So by residue theorem

$$\int_{\partial R} \frac{1}{z^5 - 1}\, dz = 2\pi i\, (\frac{1}{5} e^{2\pi i/5} + \frac{1}{5} e^{4\pi i/5}).$$

2. Let $g(z) = 2z^5 + z + 1$ and $f(z) = -6z^2$. Then Rouche's theorem on $D_1(0)$ implies that $2z^6 - 6z^2 + z + 1$ has two zeros in $D_1(0)$ and no zero on the boundary of $D_1(0)$. ($|g| < |f|$ on $D_1(0)$)

Let now $g(z) = -6z^2 + z + 1$ and $f(z) = 2z^5$. Then we apply Rouche's theorem to f,g, and $D_2(0)$, we see that $2z^6 - 6z^2 + z + 1$ has five zeros in $D_2(0)$ and no zero on the boundary of $D_2(0)$. So $2z^6 - 6z^2 + z + 1$ has 3 zeros in the annulus $1 \le |z| \le 2$.

3. (a) By the maximum modulus principle and the hypothesis on f we see that there exists $z_0 \in \mathbf{D}$ such that $f(z_0) \in \mathbf{D}$. Let $g(z) = f(z) - f(z_0)$. Then

$$|g(z) - f(z)| = |f(z_0)| < 1 = |f(z)|$$

whenever $|z| = 1$, so by Rouche's theorem f and g have the same number of zeros in $\mathbf{D}$. Since g has at least one zero in D we get the desired result.

(b) It is enough to show that if $|w_0| < 1$, then $g(z) = f(z) - w_0$ has a zero in $\mathbf{D}$. If $|z| = 1$, then

$$|g(z) - f(z)| = |w_0| < 1 = |f(z)|$$

so by Rouche's theorem, f and g have the same number of zeros in D. By part (a) f has at least one zero, so g must have at least one zero as well.

5. If u is a harmonic function on an open set Ω, then locally around every $z_0 \in \Omega$, there is a disk D such that $D \subset \Omega$. On D, u is the real part of a holomorphic function f, and f sends D to an open subset of C, so $u(\Omega)$ is the union of open sets and is therefore open.

7. Let $z = re^{i\theta}$, $0 \leq r < 1$. Then

$$\begin{aligned}
\frac{1 + re^{i\theta}}{1 - re^{i\theta}} &= (1 + z)(1 + z + z^2 + \dots) \\
&= 1 + 2\sum_{n=1}^{\infty} z^n \\
&= 1 + 2\sum_{n=1}^{\infty} r^n e^{in\theta}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
P_r(\theta) = \mathrm{Re}(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}) &= 1 + 2\sum_{n=1}^{\infty} r^n \cos n\theta \\
&= 1 + \sum_{n=1}^{\infty} r^n (e^{in\theta} + e^{-in\theta})
\end{aligned}$$