# Math 5031, Algebra I

Problem Set 2

Due: September 15 in class

1. Give an example of a group $G$ with two subgroups $K \leq H \leq G$ such that $K$ is normal in $H$ and $H$ is normal in $G$ but $K$ is not normal in $G$.

2. Assume $G_1$ and $G_2$ are two cyclic groups of orders $m_1$ and $m_2$, respectively. Show that $G_1 \times G_2$ is cyclic if and only if $m_1$ and $m_2$ are relatively prime.

3. A group $G$ is called *simple* if it has no normal subgroup other than $G$ and $\{e\}$.

(i) Show that an abelian group is simple if and only if it is finite and of prime order.

(ii) Let $f : G_1 \to G_2$ be a homomorphism which does not send every element of $G_1$ into the identity element of $G_2$. If $G_1$ is simple, show that $f$ is an injection.

4. Let $G$ be a group, and let $H_1, H_2, \ldots, H_n$ be normal subgroups of $G$ such that

(i) $G =< \bigcup_{i=1}^{n} H_i >$

(ii) For every $1 \leq j \leq n$, $H_j \cap < \bigcup_{i \neq j} H_i >= \{e\}$

Show that

$$G \cong H_1 \times H_2 \times \cdots \times H_n.$$

5. An *inner automorphism* of a group $G$ is an isomorphim $\phi : G \to G$ of the form $\phi(g) = aga^{-1}$ for some $a \in G$.

(i) Show that the inner automorphisms of $G$ form a normal subgroup of the group of all the automorphisms of $G$.

(ii) Let $Z(G) = \{x \in G | xy = yx \text{ for all } y \in G\}$. Then show that $Z(G)$ is a normal subgroup of $G$ and $G/Z(G)$ is isomorphic to the group of inner automorphisms of $G$.