

Math 5031, Algebra I

Problem Set 3

Due: September 22 in class

1. Let H be a proper subgroup of a finite group G . Show that G is not the union of all the conjugates of H .
2. Let G be a group of order p^3 , where p is prime, and G is not abelian.
 - (i) Show that $Z(G) \cong \mathbf{Z}_p$ (the cyclic group of order p), and $G/Z(G) \cong \mathbf{Z}_p \times \mathbf{Z}_p$.
 - (ii) Show that $Z(G) = G'$ (the commutator subgroup).
 - (iii) Show that every subgroup of G of order p^2 contains $Z(G)$ and is normal.
3. Assume G is a finite group of odd order, and assume H is a normal subgroup of G of order 5. Show that $H \leq Z(G)$.
4. Let G and H be two groups, and assume $\psi : G \rightarrow \text{Aut}(H)$ is a homomorphism. We define an operation on the cartesian product of G and H as follows
$$(h_1, g_1)(h_2, g_2) = (h_1\psi(g_1)h_2, g_1g_2).$$
 - (i) Show that $G \times H$ is a group under this action. This group is called the *semidirect product* of G and H .
 - (ii) Show that the natural map from H to the semidirect product of H and G sending h to (e, h) is a group homomorphism and that the image is a normal subgroup.
 - (iii) Show that if $f(g)$ is the identity automorphism of H for all $g \in G$, then the semidirect product is the direct product of H and G .
5. Show that all simple groups of order ≤ 30 must be cyclic of prime order.