Math 5031, Algebra I

Problem Set 5

Due: October 6 in class

1. Let H be a subgroup of a finite abelian group G. Show that G has a subgroup which is isomorphic to G/H.

2. Show that for every $n \geq 1$, $\mathbf{Q}/\mathbf{Z}$ has exactly one subgroup of order n and that subgroup is cyclic.

3. Let R be a ring. An element $r \in R$ is called *nilpotent* if $r^n = 0$ for some $n > 0$. Show that if R is commutative, the set of all nilpotent elements of R, denoted by $\mathrm{nil}(R)$, is an ideal in R. Is this true if R is not commutative?

4. For a prime number p, let $\mathrm{SL}(2, \mathbf{Z}_p)$ be the group of 2 by 2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbf{Z}_p$ and $ad - bc = 1$. Define

$$\mathrm{PSL}(2, \mathbf{Z}_p) := \mathrm{SL}(2, \mathbf{Z}_p) / < \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} >$$

(i) Show that $\mathrm{PSL}(2, \mathbf{Z}_2) \simeq S_3$.

(ii) Show that $\mathrm{PSL}(2, \mathbf{Z}_3) \cong A_4$.

5. Let G be an abelian group, and let $R = \mathrm{End}(G)$ be the set of all homomorphisms $\phi : G \to G$.

(i) Define addition and multiplication on R as follows. $(\phi + \psi)(g) = \phi(g) + \psi(g)$ and $\phi \cdot \psi = \phi \circ \psi$, composition of functions. Show that R is a ring.

(ii) Assume G is finitely generated. Show that R is commutative if and only if G is cyclic.