Math 5031, Algebra I

Problem Set 6

Due: October 22 in class

1. Let $f : R \to S$ be a ring homomorphism with R and S commutative.

   (i) Show that if P is a prime ideal of S, $f^{-1}(P)$ is a prime ideal of R.

   (ii) Show that if P is a maximal ideal of S, $f^{-1}(P)$ is not necessarily a maximal ideal of R.

2. Let R be a ring, and let I be an ideal of R. Let

$$\text{rad}(I) = \{x \in R | x^n \in I \text{ for some } n\}.$$

   (i) Show that $\text{rad}(I)$ is an ideal of R which contains I.

   (ii) Show that $\text{rad}(\text{rad}(I)) = \text{rad}(I)$.

$\text{rad}(I)$ is often called the *raidcal* of I.

3. Let d be an integer ≥ 1, and let R be the set of all elements of the form $a + b\sqrt{-d}$ with $a, b \in \mathbf{Z}$.

   (i) Show that R is a ring.

   (ii) Using the fact that complex conjugation is an automorphism of $\mathbf{C}$, show that complex conjugation induces an automorphism of R.

   (iii) Show that if $d \geq 2$, then the only units in R are 1 and -1.

   (iv) Show that 3, $2 + \sqrt{-5}, 2 - \sqrt{-5}$ are irreducible elements in $\mathbf{Z}[\sqrt{-5}]$. Since

$$3^2 = (2 + \sqrt{-5})(2 - \sqrt{-5}),$$

   this shows that $\mathbf{Z}[\sqrt{-5}]$ is not a UFD.

4. Show that $\mathbf{Z}[X]$ is not a PID by showing that the ideal generated by a prime p and X is not principal.

5. Let $\theta = \frac{1+\sqrt{-19}}{2}$, and define

$$R = \{a + b\theta | a, b \in \mathbf{Z}\} = \mathbf{Z}[\frac{1+\sqrt{-19}}{2}].$$

Then R is a subring of $\mathbf{C}$ (you don't need to prove this).

Define a map $\psi : R \to \mathbf{Z}+$ by

$$\psi(a + b\theta) = (a + b\theta)(\overline{a + b\theta}) = a^2 + ab + 5b^2.$$

Then we have $\psi(\alpha\beta) = \psi(\alpha)\psi(\beta)$.

(i) Using ψ, show that the only units of R are $+1$ and -1.

(ii) Using ψ, show that 2 and 3 are irreducible elements of R.

(iii) Prove that R is not a Euclidean domain using the following: Assume on the contrary that there is a map $d : R - \{0\} \to \mathbf{Z}+$ satisfying the two properties in the definition of a Euclidean domain. Let $\beta \in R - \{0, +1, -1\}$ be an element such that

$$d(\beta) = \min\{d(\alpha) | \alpha \in R, \alpha \neq 0, +1, -1\}.$$

Using the function ψ again, show that it is not possible to write $\theta = t\beta + r$ where $r = 0$, or $d(r) < d(\beta)$.

It is not hard to show that R is a PID.