Math 5031, Algebra I

Problem Set 8

Due: November 10 in class

1. Let R be a commutative ring, and let Q be a module over R. Show that the following are equivalent:

(1) If M is an R-module, if M' is a submodule of M, and if $f : M' \to Q$ is a homomorphism, then there is an extension of f to a homomorphism $M \to Q$, i.e., there is a homomorphism $h : M \to Q$ such that the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r] \arrow[d, "f"] & M \arrow[dl, "h"] \\ & Q. & \end{tikzcd}$$

(2) For any short exact sequence $0 \to M' \to M \to M'' \to 0$, we have a short exact sequence

$$0 \to \mathrm{Hom}_R(M'', Q) \to \mathrm{Hom}_R(M, Q) \to \mathrm{Hom}_R(M', Q) \to 0.$$

(3) Every exact sequence $0 \to Q \to M \to M'' \to 0$ splits.

If the above equivalent conditions are satisfied, the module Q is called an *injective* module.

2. If R is a commutative ring, and if P is a module over R which is a direct summand of a free module (so there is a module M such that $P \oplus M$ is free), then show that the functor $M \to \mathrm{Hom}_R(P, M)$ is exact.

3. **The five lemma.** Let R be a commutative ring, and consider a commutative diagram of R-modules such that every row is exact

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d, "f_1"] & M_2 \arrow[r] \arrow[d, "f_2"] & M_3 \arrow[r] \arrow[d, "f_3"] & M_4 \arrow[r] \arrow[d, "f_4"] & M_5 \arrow[d, "f_5"] \\ N_1 \arrow[r] & N_2 \arrow[r] & N_3 \arrow[r] & N_4 \arrow[r] & N_5. \end{tikzcd}$$

(i) If f_1 is surjective and f_2 and f_4 are injective, then f_3 is injective.

(ii) If f_5 is injective and f_2 and f_4 are surjective, then f_3 is surjective.

4. Let R be a PID, and let M be a finitely generated torsion module over R. For $a \in R$, set

$$M_a = \{m \in M | am = 0\}.$$

(i) Show that if $m \in M$, and if $am = 0$ and $bm = 0$, then $\gcd(a, b)m = 0$.

(ii) Show that If $a = bc$ and $\gcd(b, c) = 1$, then

$$M_a = M_b \oplus M_c.$$

(iii) Let $p \in R$ be a prime element and $n \geq 1$. Assume $x \in M_{p^n}$ is such that $p^{n-1}x \neq 0$. Let N be the submodule generated by x. Then show that for every $\bar{y} \in M_{p^n}/N$, there is a lifting $y \in M_{p^n}$ such that $a\bar{y} = 0$ if and only if $ay = 0$ for every $a \in R$.

(iv) Use part (iii) and induction to show that

$$M_{p^n} \simeq R/(p^{n_1}) \oplus \cdots \oplus R/(p^{n_s})$$

for some $n_1, \ldots, n_s$ where (p^{n_i}) is the ideal generated by p^{n_i}.