Algebra II, Spring 2017

Problem Set 1

Due: January 26 in class

1. Show that if char F $\neq 2$, then every extension of degree 2 over F is of the form $F(\sqrt{a})$ for some $a \in F$. Conclude that every degree 2 extension of F is Galois.

2. Let $\mathbf{C}(t)$ be the filed of rational functions over $\mathbf{C}$. Assume $f(t), g(t) \in \mathbf{C}[t]$ be two relatively prime polynomials, and set $\alpha = \frac{f(t)}{g(t)} \in \mathbf{C}(t)$. Assume $\alpha \notin \mathbf{C}$.

   (a) Show that the polynomial $f(x) - \alpha g(x) \in \mathbf{C}[\alpha][x]$ is irreducible. ($\mathbf{C}[\alpha]$ is the ring generated by $\mathbf{C}$ and α:

   $$\mathbf{C}[\alpha] = \{c_0 + \cdots + c_n\alpha^n \mid c_i \in \mathbf{C}, n \geq 0\}.$$

   You may need to use the fact, proved in Homework 10 last semester, that α is not algebraic over $\mathbf{C}$.)

   (b) Show that minimal polynomial of t over $\mathbf{C}(\alpha)$ is $f(x) - \alpha g(x)$.

   (c) Conclude that the degree of the extension $\mathbf{C}(\alpha) \subset \mathbf{C}(t)$ is equal to $\max\{\deg f, \deg g\}$.

3. **Automorphisms of $\mathbf{C}(t)$ over $\mathbf{C}$**: Let $\mathbf{C}(t)$ be the field of rational functions over $\mathbf{C}$. Use question 2 to show that every automorphism $\sigma \in \mathrm{Aut}(\mathbf{C}(\mathrm{t})/\mathbf{C})$ is given by $t \mapsto \frac{at+b}{ct+d}$ where $a, b, c, d \in \mathbf{C}$ and $ad - bc \neq 0$. (so $\sigma(f(t)) = f(\frac{at+b}{ct+d})$.)

4. Show that the fixed field of the automorphism $\sigma \in \mathrm{Aut}(\mathbf{C}(\mathrm{t})/\mathbf{C})$ given by $\sigma(t) = t + 1$ is $\mathbf{C}$. Conclude that if G is the subgroup of $\mathrm{Aut}(\mathbf{C}(\mathrm{t}))$ generated by σ, then $G \neq \mathrm{Aut}(\mathbf{C}(\mathrm{t})/\mathbf{C}(\mathrm{t})^{\mathrm{G}})$.