Algebra II, Spring 2017

Solutions to Problem Set 1

2 (a). Assume $f(x) - \alpha g(x) = h_1(x)h_2(x)$ where $h_1, h_2 \in \mathbf{C}[\alpha][x]$. Then h_1 is a polynomial in α and x and hence can be written as

$$h_1 = p_n(x)\alpha^n + \cdots + p_1(x)\alpha + p_0(x)$$

where the p_i are polynomials in $\mathbf{C}[x]$ and $p_n \neq 0$. Similarly $h_2(x) = q_m(x)\alpha^m + \cdots + q_1(x)\alpha + q_0(x)$ where $q_m \neq 0$. So

$$-g(x)\alpha + f(x) = (p_n(x)q_m(x))\alpha^{m+n} + \cdots + p_0(x)q_0(x).$$

Since α is not algebraic over $\mathbf{C}$ (Homework 10, Question 1, last semester), the above equality implies that the coefficients of α^i on both sides should be equal, so $m+n = 1$. Assume $n = 0$ and $m = 1$. Then $h_1 = p_0(x)$, $h_2(x) = q_1(x)\alpha + q_0(x)$, $g = -p_0q_1$, and $f = p_0q_0$. Since f and g are relatively prime, $p_0 \in \mathbf{C}$, so h_1 is a unit.

(b) Clearly t is a root of $f(x) - \alpha g(x)$, so it remain to show $f(x) - \alpha g(x)$ is irreducible. Since $\mathbf{C}(\alpha)$ is the field of fractions of $\mathbf{C}[\alpha]$, by a result from last semester this follows from part (a) if we show that $f(x) - \alpha g(x)$ is a primitive polynomial in $C[\alpha][x]$. Let $f(x) = a_nx^n + \cdots + a_0$ and $g(x) = b_mx^m + \cdots + b_0$ ($b_n, a_m \neq 0$). If $h \in \mathbf{C}[\alpha]$ is such that $h | a_i - \alpha b_i$ for all i, then $h | b_j(a_i - \alpha b_i) - b_i(a_j - \alpha b_j) = b_ja_i - b_ia_j \in \mathbf{C}$. So either there is i, j such that $h | b_ja_i - b_ia_j \neq 0$, so $h \in \mathbf{C}$ and is therefore a unit, or $b_ja_i - b_ia_j = 0$ for all i, j, so $\frac{f(t)}{g(t)} = \frac{a_m}{b_m}$. So $\alpha \in \mathbf{C}$.

3. Suppose that $\sigma(t) = \frac{f(t)}{g(t)}$ where f and g are relatively prime, and let $\alpha = \frac{f(t)}{g(t)}$. Then $\sigma(\mathbf{C}(t)) \subset \mathbf{C}(\alpha)$, so $\mathbf{C}(\alpha) = \mathbf{C}(t)$, so $[\mathbf{C}(t) : \mathbf{C}(\alpha)] = 1$. Hence by question 2, $\deg f(t), \deg g(t) \leq 1$ and $\alpha = \frac{at+b}{ct+d}$. Clearly $ad - bc \neq 0$, since otherwise $\alpha \in \mathbf{C}$.

4. If $f(t)$ and $g(t)$ are relatively prime and $\alpha = \frac{f(t)}{g(t)}$ is fixed by σ, then $\frac{f(t)}{g(t)} = \frac{f(t+1)}{g(t+1)}$, Let $c = \frac{f(0)}{g(0)}$. Then

$$c = \frac{f(0)}{g(0)} = \frac{f(1)}{g(1)} = \frac{f(2)}{g(2)} = \ldots$$

(whenever the denominator is not zero.), so $f(t) - cg(t)$ has infinitely many zeros and so is the zero polynomial, so $\alpha \in \mathbf{C}$.