Algebra II, Spring 2017

Problem Set 2

Due: February 3

1. Let E be the splitting field of $f(x) = x^4 - 2 \in \mathbf{Q}[x]$ and G the Galois group of $E/\mathbf{Q}$. Describe the subfields of E corresponding to the following subgroups of G: $< gh >$, $< gh^3 >$, and $< g, h^2 >$ (If $\rho = \sqrt[4]{2}$, then g is the automorphism which fixes ρ and sends i to $-i$, and h is the automorphism which fixes i and sends ρ to $i\rho$.)

2. Find the Galois group of the following polynomials.

   (a) $x^3 + x^2 - 2x - 1$ over $\mathbf{Q}$

   (b) $x^3 - 10$ over $\mathbf{Q}(\sqrt{2})$

3. Suppose that $f(x) \in F[x]$ is an irreducible separable polynomial and E is the splitting field of f. Then show that the Galois group of E/F acts transitively on the roots of $f(x)$.

4. Let E/F be a finite Galois extension and $G = \mathrm{Gal(E/F)}$. If $H_1 \leq H_2 \leq G$, then show that $H_1 \trianglelefteq H_2$ if and only if E^{H_1} is a normal extension of E^{H_2}.

5. Let $f(x) = x^4 + ax^2 + b$ be an irreducible polynomial over $\mathbf{Q}$, with roots $\pm\alpha, \pm\beta$, and splitting field E.

   (a) Show that $\mathrm{Gal(E/}\mathbf{Q})$ is isomorphic to a subgroup of the Dihedral group of order 8, D_8, and is therefore isomorphic to $\mathbf{Z}_4$, $\mathbf{Z}_2 \times \mathbf{Z}_2$, or D_8.

   (b) Show that if $\alpha\beta \in \mathbf{Q}$, then $G = \mathbf{Z}_2 \times \mathbf{Z}_2$.

   (c) Show that $\frac{\alpha}{\beta} - \frac{\beta}{\alpha} \in \mathbf{Q}$ if and only if $G = \mathbf{Z}_4$.

6. Show that if the Galois group of an irreducible polynomial of degree 3 in $\mathbf{Q}[x]$ is $\mathbf{Z}_3$, then all the roots of the polynomial are real.