Algebra II, Spring 2017

Problem Set 3

Due: February 14 in class

1. Let F be a finite field and let $F \subset E$ be a finite extension. Show that the extension E/F is Galois and its Galois group is cyclic.

2. Let p be a prime number, and let $E = \mathbf{F}_p(x, y)$ and $F = \mathbf{F}_p(x^p, y^p)$. Show that E/F is a finite extension which is not generated by one element over F.

3. Let $f(x) \in F[x]$ be an irreducible polynomial of degree 5 whose discriminant is a non-zero square in F. Find all possible Galois groups for its splitting field.

4. Let p be a prime number and $n \geq 2$ an integer.

(a) Show that there is an irreducible polynomial of degree n in $\mathbf{F}_p[x]$.

(b) Show that there is an irreducible polynomial of degree n in $\mathbf{Q}[x]$ with exactly $n-2$ real roots. (hint: let $g(x) = a_n x^n + \cdots + a_0$ be a reducible such polynomial. Since the roots of a polynomial depend continuously on its coefficients, a small perturbation of $g(x)$ has also $n-2$ real roots: for sufficiently small $\epsilon \in \mathbf{Q}$, $h(x) = b_n x^n + \cdots + b_0 \in \mathbf{Q}[x]$ has also $n-2$ real roots if $|b_i - a_i| < \epsilon$ for all i. Use this fact and the polynomial in part (a) to construct the desired polynomial.)

(c) Let $f(x) \in \mathbf{Q}[x]$ be an irreducible polynomial of degree p. Suppose that $f(x)$ has exactly $p-2$ real roots. Show that the Galois group of the splitting field of $f(x)$ is S_p.