Algebra II, Spring 2017

Problem Set 4

Due: February 28 in class

1. **Trace map:** Let E/F be a finite extension. For $\alpha \in E$, the *trace* of α, denoted by $T(\alpha)$, is defined as the trace of the F-linear map

$$L_\alpha : E \to E, \qquad L_\alpha(x) = \alpha x.$$

So for every $\alpha \in E$, $T(\alpha) \in F$.

(a) Show that if E/F is a finite Galois extension with Galois group G, then

$$T(\alpha) = \sum_{\sigma \in G} \sigma(\alpha).$$

(b) Use independence of characters to show that the map T is not identically zero.

2. Show that if E/F is a Galois extension with cyclic group $G =< \sigma >$, then

$$\text{Kernel}(T) = \{\alpha \in E \mid \alpha = \beta - \sigma(\beta) \text{ for some } \beta \in E\}.$$

(This is the additive version of Hilbert's Theorem 90).

3. Let F be a field of characteristic p.

(a) Let $f(x) = x^p - x - c$ be a polynomial over F, and let E be the splitting field of $f(x)$. If α is a root of $f(x)$ in E, then show that every root of $f(x)$ is of the form $\alpha + j$, for $0 \leq j < p$.

(b) Assume L/F is a Galois extension of order p with cyclic Galois group G. Use Problem 2 to prove that $L = F(\alpha)$ for some $\alpha \in L$ such that α is a root of a polynomial of the form $x^p - x - c \in F[x]$.

4. This exercise proves a reduction step we took when we showed the Galois group of a polynomial solvable by radicals is a solvable group.

Let F be a field of characteristic zero and let $f(x)$ be a polynomial over F. Let

$$F = F_0 \subset F_1 \subset \cdots \subset F_m$$

be a tower of fields such that

- $F_i = F_{i-1}(\alpha_i)$, with $\alpha_i^{n_i} \in F_{i-1}$ for some α_i and $n_i \geq 1$.
- $f(x)$ splits in F_m.

Then show that there is such a tower with the additional property that F_m is the splitting field of a polynomial over F. (Hint: Let f_i be the minimal polynomial of α_i over F, and consider the splitting field of $f_1 \ldots f_m$.)

5. Use Hilbert Theorem 90 to find the rational solutions of the equation $x^2 + dy^2 = 1$ where d is a positive integer.