Algebra II, Spring 2017

Problem Set 5

Due: March 9 in class

1. If $F \subset E \subset K$ is a tower of fields such that the transcendence degree of K over F is finite, then show

$$\mathrm{tr.\ deg.\ K/F = tr.\ deg.\ K/E + tr.\ deg.\ E/F.}$$

2. Let $F \subset E \subset K$ be a tower of fields.

   (a) If E is algebraic over F and A is a subset of K which is algebraically independent over F, then show that A is algebraically independent over E.

   (b) Use part (a) to show that if K is finitely generated over F, then E is finitely generated over F.

3. Let $f(x) = x^4 + 2x^2 + x + 3 \in \mathbf{Q}[x]$. Show that f is irreducible with no repeated roots mod 2, and it has an irreducible factor of degree 3 mod 3. Conclude that the Galois group of $f(x)$ is S_4.

4. Let $f(x) = x^6 + 22x^5 - 9x^4 + 12x^3 - 37x^2 - 29x - 15 \in Q[x]$. Show that the Galois group of $f(x)$ is S_6 by looking at $f(x)$ mod 2 and mod 5. (you can use the following fact without proving it: a subgroup of S_6 which contains a 6-cycle and a transposition is S_6.)

5. If F is a finite field, and $\bar{F}$ is its algebraic closure, then show that $\mathrm{Aut(\bar{F}/F)}$ is abelian.