Algebra II, Spring 2017

Problem Set 6

Due: March 28 in class

In Questions 1-3 A is an integral domain, K is the quotient field of A, and we assume A is integrally closed in K. The field L is a finite extension of K and B is the integral closure of A in L.

1. Show that for any $b \in B$, the minimal polynomial of b over K is in $A[x]$.

2. Prove that L is the quotient field of B. (if $l \in L$, then show that there is $c \in A$ such that $c\,l$ is integral over A by looking at the minimal polynomial of l over K.)

3. Suppose that L is the splitting field of a polynomial $f(x) \in A[x]$ with leading coefficient 1. Show that if $\mathfrak{q}$ is a maximal ideal of B and $\mathfrak{p} = \mathfrak{q} \cap A$, then $B/\mathfrak{q}$ is the splitting field of $\bar{f} \in A/\mathfrak{p}\,[x]$.

4. Let A be an integral domain. A is said to be *integrally closed* if it is integrally closed in its quotient field.

   (a) Show $A = \cap_{\mathfrak{p}} A_{\mathfrak{p}}$ (as subsets of the quotient field of A) where the intersection is over all prime ideals of A, and $A_{\mathfrak{p}} = S^{-1}A, S = A \setminus \mathfrak{p}$. (Consider an element $c \in K$ and the ideal $I = \{a \in A \mid ac \in A\}$. Show that if $c \notin A$, then $I \neq A$ and is therefore contained in a maximal ideal $\mathfrak{m}$.)

   (b) Show A is integrally closed if and only if $A_{\mathfrak{p}}$ is integrally closed for every prime ideal $\mathfrak{p}$.