Algebra II, Spring 2017

Solutions to Problem Set 6

1. We need to assume that L is a Galois extension here. Let $f(x) \in K[x]$ be the minimal polynomial of b over K. Let $b = b_1, \ldots, b_n$ be the roots of $f(x)$ in L. Then since the extension is Galois and $f(x)$ is irreducible, Gal(L/K) acts transitively on the roots of f, so for each i, there is an element σ of the Galois group such that $\sigma(b) = b_i$, so b_i is also integral over A. But if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then for each m, $1 \leq m \leq n$,

$$a_{n-m} = (-1)^m \sum_{j_1 < \cdots < j_m} b_{j_1} \ldots b_{j_m}$$

and since integrally closed elements over A form a ring, the coefficients of f should be integral over A and since A is integrally closed in K, they are in A.

2. Pick $l \in L$. We show that there is $c \in A$ such that $cl = b \in B$, so $l = \frac{b}{a}$. Let

$$f(x) = x^n + \frac{a_{n-1}}{c_{n-1}} x^{n-1} + \cdots + \frac{a_0}{c_0}$$

be the minimal polynomial of l over K. Multiplying by c^n where $c = c_0 \ldots c_{n-1}$, we see that

$$c^n l^n + a'_{n-1} c^{n-1} l^{n-1} + \cdots + a'_1 cl + a'_0$$

where $a'_i = c^{n-i} \frac{a_i}{c_i} \in A$ for $0 \leq i \leq n-1$. So cl is a root of the polynomial

$$x^n + a'_{n-1} x^{n-1} + \cdots + a'_0 \in A[x].$$

Therefore cl is integral over A so it is in B.

3. Let the roots of f be $u_1, \ldots u_n \in L$, so $f(x) = (x - u_1) \ldots (x - u_n)$. Each U_i is integral over A, so $u_i \in B$ for every i. So we have

$$\bar{f}(x) = (x - \bar{u}_1) \ldots (x - \bar{u}_n).$$

where $\bar{u}_i$ is the image of u_i in $B/\mathfrak{q}$. So $\bar{f}$ splits in $B/\mathfrak{q}$. It remains to show the roots of $\bar{f}$ generate $B/\mathfrak{q}$. Every element of $B/\mathfrak{q}$ is of the form $\bar{b} = b + \mathfrak{q}$ for some $b \in B$. Since $b \in L$, and L is generated by the u_i over K, we have

$$b = \frac{g(u_1, \dots, u_n)}{h(u_1, \dots, u_n)}$$

for some $g, h \in k[x_1, \dots, x_n]$. Multiplying by common denominators, we can assume $g, h \in A[x]$. So

$$\bar{b} = \frac{g(\bar{u_1}, \dots, \bar{u}_n)}{h(\bar{u}_1, \dots, \bar{u}_n)}.$$

4.(a) Clearly $A \subset \cap_{\mathfrak{p}} A_{\mathfrak{p}}$. Conversely, suppose an element $c = \frac{x}{y}$ of K is in $\cap_{\mathfrak{p}} A_{\mathfrak{p}}$, and let $I = \{a \in A \mid ac \in A\}$. Then if $c \notin A$, $1 \notin I$, so $I \neq A$, so there is maximal ideal $\mathfrak{m}$ of A containing I. By our assumption $c \in A_{\mathfrak{m}}$, so $c = \frac{x}{y} = \frac{a}{b}$ where $b \notin \mathfrak{m}$. But then $bc = b\frac{x}{y} = a \in A$, so by definition of I, $b \in I$. This contradicts the fact that $\mathfrak{m}$ contains I. So c has to be in A.

(b) Let K be the quotient field of A. Then for every prime ideal $\mathfrak{p}$ of A,

$$A \subset A_{\mathfrak{p}} \subset K.$$

So K is the quotient field of $A_{\mathfrak{p}}$ for every $\mathfrak{p}$. Assume that $A_{\mathfrak{p}}$ is closed for every $\mathfrak{p}$, and let $c \in K$ be integral over A. The for every $\mathfrak{p}$, c is integral over $A_{\mathfrak{p}}$, so $s \in A_{\mathfrak{p}}$, so $c \in \cap_{\mathfrak{p}} A_{\mathfrak{p}} = A$.

Conversely, suppose that A is integrally closed, and $l \in K$ is integral over $A_{\mathfrak{p}}$. So l is the root of a polynomial

$$f(x) = x^n + \frac{a_{n-1}}{b_{n-1}} x^{n-1} + \cdots + \frac{a_0}{b_0}$$

where the b_i are not in $\mathfrak{p}$. Let $b = b_0 \dots b_{n-1}$. Then $b \notin \mathfrak{p}$. Multiplying the above polynomial by b^n, we see that bl is the root of a monic polynomial in $A[x]$, so $bl \in A$, so $l = \frac{a}{b} \in A_{\mathfrak{p}}$. So $A_{\mathfrak{p}}$ is integrally closed.