Algebra II, Spring 2017

Problem Set 7

Due: April 6 in class

1. Assume that A is a commutative ring.

   (a) Generalize Question 5, from Homework 5 last semester, to show the following: If S is a multiplicative subset of A and I is an ideal of A such that $I \cap S = \emptyset$, then there is a prime ideal $\mathfrak{p}$ containing I such that $p \cap S = \emptyset$.

   (b) Conclude that $rad(I)$ is the intersection of all prime ideals which contains I.

2. Show that if I is an ideal such that $rad(I)$ is a maximal ideal, then I is a primary ideal. (use Question 1)

3. Give an example of an ideal I such that $rad(I)$ is a prime ideal, but I is not primary.

4. Show that if A is a Noetherian ring and S is a multiplicative subset of A, then $S^{-1}A$ is Noetherian.

5. In this problem you will see an example of a polynomial $f(x) \in \mathbf{Z}[x]$ such that $f(x)$ is reducible mod p for every prime p, but $f(x)$ is irreducible in $Q[x]$.

   (a) Show that $x^4 - 10x^2 + 1$ is irreducible in $\mathbf{Q}[x]$. (Hint: since $\mathbf{Z}$ is integrally closed, every rational root of f has to be an integer. Show that the f has no integer roots. Then argue that there are not rational numbers a, b, c such that

   $$x^4 - 10x^2 + 1 = (x^2 + ax + b)(x^2 - ax + c).$$

   (b) Show that the Galois group of $f(x)$ over $\mathbf{Q}[x]$ is $\mathbf{Z}_2 \times \mathbf{Z}_2$.

   (c) Show that $f(x)$ mod p is reducible in $\mathbf{F}_p[x]$ for every prime number p.