Algebra II, Spring 2017

Problem Set 8

Due: April 18 in class

1. Let $\mathfrak{p}_1 = (x, y), \mathfrak{p}_2 = (x, z)$, and $\mathfrak{m} = (x, y, z)$ in $k[x, y, z]$ where k is an arbitrary field. Show that

$$\mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{m}^2$$

is a minimal primary decomposition of the ideal $\mathfrak{p}_1\mathfrak{p}_2$.

2. Show that if A is a Dedekind domain and $\mathfrak{p}$ is a prime ideal of A, then $A_\mathfrak{p}$ is also a Dedekind domain.

3. Assume A is a Noetherian local domain with maximal ideal $\{0\} \neq \mathfrak{m}$. Let K be the quotient field of A, and assume $t \in K$ is not integral over A. Then prove that $t\,\mathfrak{m} \not\subset \mathfrak{m}$. (Hint: use the fact that $\mathfrak{m}$ is a finitely generated ideal and mimic the proof of the fact if $A[t]$ is a finitely generated A-module, then t is integral over A.)

4. Let $L = \mathbf{Q}(\sqrt{d})$ be a quadratic extension of $\mathbf{Q}$ where d is a square-free odd integer, and let B be the integral closure of $\mathbf{Z}$ in L. Show that

$$B = \begin{cases} \mathbf{Z}[\sqrt{d}] & \text{if } d \equiv 3 \mod 4 \\ \mathbf{Z}[\frac{1+\sqrt{d}}{2}] & \text{if } d \equiv 1 \mod 4 \end{cases}$$

5. Let A be a Dedekind domain with finitely many prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$.

(a) Show that every prime ideal $\mathfrak{p}_i$ is principal. (Hint: use the Chinese Remainder Theorem to show there is $x \in A$ such that $x \notin \mathfrak{p}_j \;\; i \neq j$, and $x \in \mathfrak{p}_i \setminus \mathfrak{p}_i^2$. Then look at the factorization of (x) into product of prime ideals, to show $(x) = \mathfrak{p}_i$)

(b) Conclude that every ideal in A is principal.