Algebra II, Spring 2017

Solutions to Problem Set 8

2. We have seen in Homework 7 that localization of Noetherian rings are Noetherian. Also Homework 6, Question 4 shows if A is integrally closed, then $A_{\mathfrak{p}}$ is integrally closed. It remains to show every non-zero prime ideal in $A_{\mathfrak{p}}$ is maximal, but this is clear since the prime ideals of $A_{\mathfrak{p}}$ are in one-to-one correspondence to prime ideals of A contained in $\mathfrak{p}$, so $A_{\mathfrak{p}}$ has only one non-zero prime ideal and that is $\mathfrak{p}A_{\mathfrak{p}}$.

3. Assume $t\mathfrak{m} \subset \mathfrak{m}$. Since A is Noetherian, $\mathfrak{m}$ is finitely generated. Let $\mathfrak{m} = (x_1, \ldots, x_n)$. Since $tx_i \in \mathfrak{m}$ by our assumption, $tx_i = \alpha_{i1}x_1 + \ldots \alpha_{in}x_n$ for some $\alpha_{ij} \in A$. If M is the n by n matrix whose entries are α_{ij}, then $M - tI_{n\times n}$ has the vector $[x_1 \ \ldots \ x_n]$ in its kernel, so $\det(M - xI)$ vanishes at t, and therefore, t is integral over A

4. If $d \equiv 3 \bmod 4$, then this is identical to the case of $\sqrt{-5}$ we did in class. So we assume $d \equiv 1 \bmod 4$, and $d = 4d' + 1$. Elements of L are of the form $\alpha = \frac{a}{b} + \frac{e}{f}\sqrt{d}$ for rational numbers $\frac{a}{b}$ and $\frac{e}{f}$. We can assume $\gcd(a, b) = \gcd(e, d) = 1$. Assume α is integral over $\mathbf{Z}$. Let $f(x) \in \mathbf{Z}[x]$ be a polynomial with leading coefficient 1 which vanishes at α. Then $\alpha' = \frac{a}{b} - \frac{e}{f}\sqrt{d}$ is a root of f too, so it is integral over $\mathbf{Z}$. Since integral elements form a ring, $\alpha + \alpha' = \frac{2a}{b}$ and $\alpha\alpha' = \frac{a^2}{b^2} + \frac{e^2 d}{f^2}$ are also integral over $\mathbf{Z}$. Since the only elements of $\mathbf{Q}$ which are integral over $\mathbf{Z}$ are integers, $b = 1$ or 2, and $f = 1$ or 2 (since d is square-free and odd, and e and f are coprime). If $b = 1$, then since $\alpha\alpha'$ is integral, $\frac{e^2 d}{f^2} \in \mathbf{Z}$, and since d is square free, $f = 1$. And similarly if $f = 1$, b has to be 1 too. If $f = b = 2$, then $4|a^2 + de^2$, so a and e are either both even or both odd, so $a - e$ is even. Therefore α is either of the form $a + e\sqrt{d}$ or $\frac{a+e\sqrt{d}}{2}$ with $a - e$ even, hence $\alpha \in \mathbf{Z}[\frac{1+\sqrt{d}}{2}]$.

On there other hand $\alpha = \frac{1+\sqrt{d}}{2}$ is integral over $\mathbf{Z}$: $\alpha^2 - \alpha - d' = \frac{1+d+2\sqrt{d}-2-2\sqrt{d}}{4} - d' = 0$.

5. (a) We first show that $\mathfrak{p}_i \neq \mathfrak{p}_i^2$. If we localize A at $\mathfrak{p}_i$, then $A_{\mathfrak{p}_i}$ is a local ring which

is a Dedekind domain (by Question 2). We proved in class that its unique maximal ideal $\mathfrak{m} = \mathfrak{p}_i A_{\mathfrak{p}_i}$ is generated by one element u. If $\mathfrak{p}_i = \mathfrak{p}_i^2$, then $(u) \in \mathfrak{m} = \mathfrak{m}^2 = (u^2)$. But then $u = xu^2$ for some $x \in A_{\mathfrak{p}_i}$, so $u(1 - xu) = 0$, so u is a unit, so $\mathfrak{m}$ is not a maximal ideal, a contradiction. So pick an element $a \in \mathfrak{p}_i$ which is not in $\mathfrak{p}_i^2$. By the Chinese reminder theorem there is $x \in A$ such that $x \equiv 1 \mod \mathfrak{p}_j,\ j \neq i$, and $x \equiv a \mod \mathfrak{p}_i^2$. (note that $\mathfrak{p}_i^2$ and $\mathfrak{p}_j$ are coprime for every $j \neq i$.) The choice of x shows that $x \notin \mathfrak{p}_j, j \neq i$, and $x - a \in \mathfrak{p}_i^2 \subset \mathfrak{p}_i$, so $x \in \mathfrak{p}_i$, but $x \notin \mathfrak{p}_i^2$.

Now look at the primary decomposition of (x)

$$(x) = \mathfrak{p}_1^{n_1} \dots \mathfrak{p}_k^{n_k}.$$

If $n_j > 0$ for $j \neq i$, then $x \in \mathfrak{p}_j$, so $n_j = 0$ for all $j \neq i$. Since $x \notin \mathfrak{p}_i^2$, $n_i = 1$, so $(x) = \mathfrak{p}_i$.

(b) Every ideal $\mathfrak{a}$ has a primary decomposition $\mathfrak{a} = \mathfrak{p}_1^{n_1} \dots \mathfrak{p}_k^{n_k}$. If $\mathfrak{p}_i$ is generated by x_i, then $\mathfrak{a}$ is generated by $x_1^{n_1} \dots x_k^{n_k}$.