Algebra II, Spring 2017

Solutions to Problem Set 9

1. (a) The polynomial ring is a UFD, so the irreducible polynomial f generates a prime ideal, and therefore $V(\{f\})$ is irreducible.

(b) $V(f)$ has two distinct points, so it is not irreducible. If

$$f(x,y) = g(x,y)h(x,y),$$

then for every $0 \neq a \in \mathbf{R}$,

$$f(x,a) = g(x,a)h(x,a),$$

and $f(x,a)$ has no roots, so $g(x,a)$ and $h(x,a)$ have not roots in $\mathbf{R}$. This implies that they both should be of degree 2 in $\mathbf{R}[x]$ for every non-zero a. But then $g(x,a)$ would be a polynomial of degree 2 whose coefficients are functions in a, so the discriminant would be also a function in a, and therefore it has to be either a negative constant number or positive for infinitely many a. The same is true for $h(x,q)$. This is a contradiction since $f(x,a)$ has a root only when $a = 0$.

2. Let $\mathfrak{a} \subset k[x_1, \ldots, x_n]$ be an ideal, and let $\mathfrak{b}$ be the intersection of all the maximal ideals containing $\mathfrak{a}$. We want to show $\sqrt{\mathfrak{a}} = \mathfrak{b}$. Every maximal ideal is radical, so clearly $\sqrt{\mathfrak{a}} \subset \mathfrak{b}$. Conversely, let $f \in \mathfrak{b}$. By Hilbert's Nullstellensatz, we know $I(V(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, so it is enough to show $f \in I(V(\mathfrak{a}))$, or equivalently, for every $p \in V(\mathfrak{a})$, $f(p) = 0$. This is true because the ideal $I(\{p\})$ is a maximal ideal which contains $\mathfrak{a}$ ($p \in V(\mathfrak{a})$, so $I(\{p\}) \supseteq I(V(\mathfrak{a})) \supseteq \mathfrak{a}$), so by our assumption $f \in I(\{p\})$, which means $f(p) = 0$.

3. (a) Let $V_1 \supseteq V_2 \supseteq \ldots$ be a descending sequence of closed subsets of V. Then each V_i is an algebraic closed subset of $\mathbf{A}_k^n$, and

$$I(V_1) \subseteq I(V_2) \subseteq \ldots$$

is an ascending chain of ideals which should stabilize since the polynomial ring is Noetherian. So there is m such that

$$I(V_m) = I(V_{m+1}) = \ldots,$$

But since the V_i are closed algebraic subsets $V(I(V_i)) = V_i$, so by taking V, we see

$$V_m = V_{m+1} = \dots.$$

(b) Let $U_i, i \in I$, be an open cover of V, and assume to the contrary there is no finite subcover. Let U_{i_1} be one of the open subsets in the cover. There should be $p_1 \in V$ which is not in U_{i_1}. Pick U_{i_2} which contains p_1. There is a point $p_2 \in V$ which is not in $U_{i_1} \cap U_{i_2}$. Pick U_{i_3} such that it contains p_2. Continuing, we get an ascending chain of open subsets

$$U_{i_1} \underset{\neq}{\subsetneq} U_{i_1} \cap U_{i_2} \underset{\neq}{\subsetneq} \dots$$

The complement of this chain gives an descending chain of closed subset which does not stabilize, a contradiction.

4. If $x^2 - yz = xz - x = 0$, then $x = y = 0$ or $x = z = 0$ or $z = 1, x^2 = y$. Clearly $V(x, y)$ is irreducible (the ideal generated by x, y is prime.). similarly $V(x, z)$ is irreducible. Finally $V(z-1, x^2-y)$ is irreducible because if $I = (z-1, x^2-y)$, then $k[x, y, z]/I$ is isomorphic to a polynomial ring in one variable via the homomorphism

$$k[x, y, z]/I \to k[t]$$

$$x \mapsto t, y \mapsto t^2, z \mapsto 1, \quad \text{so} \quad f(x, y, z) \mapsto f(t, t^2, 1)$$

5. (a) If $f \in I(X_1 \cup X_2)$, then f vanishes on both X_1 and X_2, so $f \in I(X_1) \cap I(X_2)$, and conversely, if f vanishes on both X_1 and X_2, then it vanishes on their union.

(b) One direction is obvious: $X_1 \cap X_2 \subseteq X_i$, $i = 1, 2$, so $I(X_i) \subseteq I(X_1 \cap X_2)$, so $I(X_1) + I(X_2) \subseteq I(X_1 \cap X_2)$, and since X_1, X_2 are closed algebraic subsets, so is $X_1 \cap X_2$, hence $I(X_1 \cap X_2)$ is a radical ideal. Therefore $\sqrt{I(X_1) + I(X_2)} \subseteq I(X_1 \cap X_2)$.

For the other direction, we use Hilbert's Nullstellensatz. Let $J = I(X_1) + I(X_2)$. By Nullstellensatz, $I(V(J)) = \sqrt{J}$. So we need to show $I(X_1 \cap X_2) \subseteq I(V(J))$. To show this, it is enough to show $V(J) \subseteq X_1 \cap X_2$. This is true since if $p \in V(J)$, then every polynomial in $I(X_1) + I(X_2)$ vanishes at p, in particular every polynomial in $I(X_1)$ and every polynomial in $I(X_2)$ vanishes at p, so $p \in V(I(X_1)) \cap V(I(X_2))$. We know $V(I(X_i)) = X_i$, so $p \in X_1 \cap X_2$.

6. By Question 5 (a), the ideal is $I = (y) \cap (x-1, y-1)$. We show $I = (y(x-1), y(y-1))$ The reason is that if $f(x, y) \in I$, then $f(x, y) = yg(x, y)$ for some g, and $yg(x, y) \in \mathfrak{m} = (x-1, y-1)$. But $\mathfrak{m}$ is a maximal ideal and $y \notin \mathfrak{m}$ (the function y does not vanish on $(1, 1)$), so $g(x, y) \in \mathfrak{m}$, so $f \in (y(x-1), y(y-1))$.