Roya Beheshti Zavareh **math233fall2008**

WeBWorK assignment number set13 is due : 12/05/2008 at 10:00pm CST.

The

(* replace with url for the course home page *)

for the course contains the syllabus, grading policy and other information.

This file is /conf/snippets/setHeader.pg you can use it as a model for creating files which introduce each problem set.

The primary purpose of WeBWorK is to let you know that you are getting the correct answer or to alert you if you are making some kind of mistake. Usually you can attempt a problem as many times as you want before the due date. However, if you are having trouble figuring out your error, you should consult the book, or ask a fellow student, one of the TA's or your professor for help. Don't spend a lot of time guessing – it's not very efficient or effective.

Give 4 or 5 significant digits for (floating point) numerical answers. For most problems when entering numerical answers, you can if you wish enter elementary expressions such as $2 \wedge 3$ instead of 8, $sin(3 * pi/2)$instead of -1, $e \wedge (ln(2))$ instead of 2, $(2+tan(3))*(4-sin(5)) \wedge 6-7/8$ instead of 27620.3413, etc. Here's the **list of the functions** which WeBWorK understands.

You can use the Feedback button on each problem page to send e-mail to the professors.

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1. **(1 pt) Library/Dartmouth/setMTWCh7S1/problem_2.pg**

Let C be the positively oriented square with vertices $(0,0)$, $(2,0)$, $(2,2)$, $(0,2)$. Use Green's Theorem to evaluate the line integral $\int_C 8y^2x\,dx + x^2y\,dy$.

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2. **(1 pt) Library/Dartmouth/setMTWCh7S1/problem_1.pg**

Let C be the positively oriented circle $x^2+y^2=1$. Use Green's Theorem to evaluate the line integral $\int_C 18y\,dx + 8x\,dy$.

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3. **(1 pt) Library/272/setStewart16_4/ur_vc_12_8.pg**

A)

Use Green's theorem to compute the area inside the ellipse $\frac{x^2}{10^2}+\frac{y^2}{19^2}=1$.

Use the fact that the area can be written as

$$\iint_D dx\,dy = \frac{1}{2}\int_{\partial D} -y\,dx + x\,dy\,.$$

Hint: $x(t) = 10\cos(t)$.

The area is __________.

B)

Find a parametrization of the curve $x^{2/3}+y^{2/3}=5^{2/3}$ and use it to compute the area of the interior.

Hint: $x(t) = 5\cos^3(t)$.

____

From Rogawski ET section 16.3, exercise 17.

Determine whether the vector field is conservative and, if so, find the general potential function.

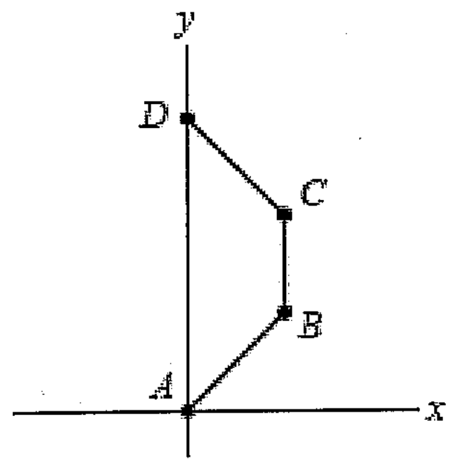
$$\mathbf{F} = \langle \cos z, 2y^{17}, -x\sin z\rangle$$

$\varphi =$ __________ $+c$

Note: if the vector field is not conservative, write "DNE".

From Rogawski ET section 17.1, exercise 23.

Evaluate $I = \int_C (\sin x + 4y)\,dx + (12x+y)\,dy$ for the non-closed path $ABCD$ in the figure.



$A = (0,0)$, $B=(1,1)$, $C=(1,2)$, $D=(0,3)$

$I =$ __________

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6. **(1 pt) Library/Dartmouth/setMTWCh6S3/problem_1.pg**

Write down the iterated integral which expresses the surface area of $z = y^3\cos^5 x$ over the triangle with vertices (-1,1), (1,1), (0,2):

$$\int_a^b \int_{f(y)}^{g(y)} \sqrt{h(x,y)}\,dx\,dy$$

$a =$ __________

$b =$ __________

$f(y) =$ __________

$g(y) =$ __________

$h(x,y) =$ __________

7. (1 pt) Library/ASU-topics/setCalculus/stef16_6p4.pg

A sphere of radius 2 is centered at the origin.
It may be viewed as a parametrized surface: $\mathbf{r}(\theta,\phi) = (2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi)$, a level surface of the function $f(x,y,z) = x^2+y^2+z^2$, or as the graph of the function $g(x,y) = \sqrt{4-x^2-y^2}$.
Consider the sphere at the point $(1.00000, 1.00000, 1.41421)$ (corresponding to $(\theta,\phi) = (\pi/4, \pi/4)$).

A) Find the normal vector $\mathbf{r}_\theta \times \mathbf{r}_\phi$ at the given point:
(________, ________, ________)

B) Find the gradient of f at the indicated point:
(________, ________, ________)

They should be parallel ....

8. (1 pt) Library/272/setStewart16_6/problem_2.pg
Find the surface area of that part of the plane $8x+10y+z=8$ that lies inside the elliptic cylinder $\frac{x^2}{16}+\frac{y^2}{9}=1$

Surface Area = ________

From Rogawski ET section 16.4, exercise 3.
Show that $\Phi(u,v) = (3u+7, u-v, 5u+v)$ parametrizes the plane $2x-y-z=14$. Then:
(a) Calculate $\mathbf{T}_u$, $\mathbf{T}_v$, and $\mathbf{n}(u,v)$.
(b) Find the area of $\mathcal{S} = \Phi(\mathcal{D})$, where $\mathcal{D} = (u,v) : 0 \le u \le 8, 0 \le v \le 7$.
(c) Express $f(x,y,z) = yz$ in terms of u and v and evaluate $\iint_S f(x,y,z)\,dS$.
(a) $\mathbf{T}_u$ = ________ , $\mathbf{T}_v$ = ________ , $\mathbf{n}(u,v)$ = ________
(b) $Area(S)$ = ________
(c) $\iint_S f(x,y,z)\,dS$ = ________

10. (1 pt) Library/Rochester/setVectorCalculus3/ur_vc_13_2.pg
Find the surface area of the part of the sphere $x^2+y^2+z^2=100$ that lies above the cone $z=\sqrt{x^2+y^2}$

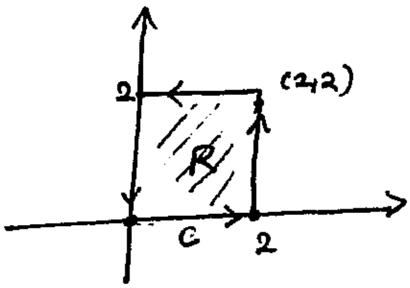
____

From Rogawski ET section 16.4, exercise 27.
Calculate $\iint_S f(x,y,z)\,dS$ For

$$y = 4-z^2, \qquad 0 \le x,z \le 6; \qquad f(x,y,z) = z$$

$\iint_S f(x,y,z)\,dS$ = ________

1.

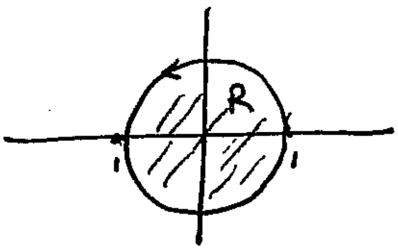


Green's Theorem implies: $\oint_C 8y^2x\,dx + x^2y\,dy = \iint_R \frac{\partial}{\partial x}(x^2y) - \frac{\partial}{\partial y}(8y^2x)\,dA$

$= \iint_R 2xy - 16yx\,dA = \iint_R -14xy\,dA = \int_0^2\int_0^2 -14xy\,dy\,dx$

$= \int_0^2 -7xy^2\Big|_0^2 dx = \int_0^2 -28x\,dx = -14x^2\Big|_0^2 = -56$

2.



C: positively oriented unit circle.

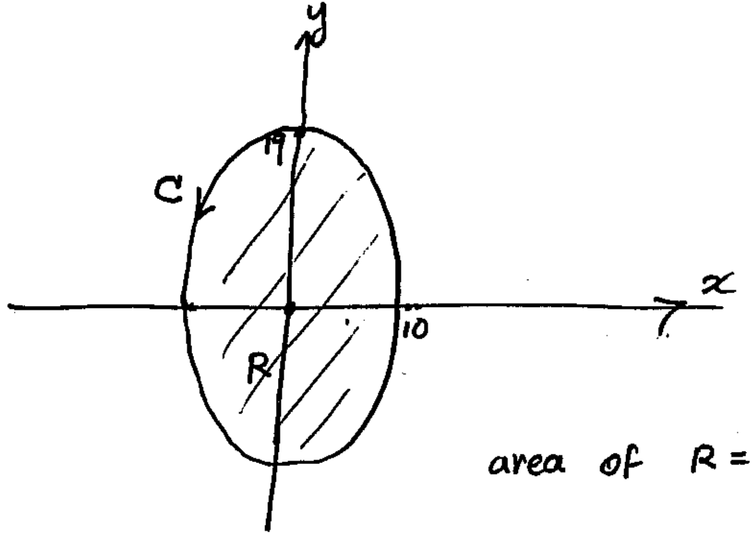
Green's Theorem : $\oint_C 18y\,dx + 8x\,dy = \iint_R \frac{\partial}{\partial x}(8x) - \frac{\partial}{\partial y}(18y)\,dA$

$= \iint_R (8-18)\,dA = \iint_R -10\,dA = -10$ area of $R = -10\times\pi$

3. A) A parametrization for ellipse $\frac{x^2}{10^2}+\frac{y^2}{19^2}=1$ is

given by $\vec{r}(t) = \langle 10\cos t, 19\sin t\rangle \quad 0\le t\le 2\pi$

$\frac{d\vec{r}}{dt}(t) = \langle -10\sin t, 19\cos t\rangle$



area of $R = \iint_R 1 \, dA$

Green's Theorem: $\text{area} = \frac{1}{2}\oint_C -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} \langle -19\sin t, 10\cos t\rangle$

$\cdot \langle -10\sin t, 19\cos t\rangle dt = \frac{1}{2}\int_0^{2\pi} 190\sin^2 t + 190\cos^2 t\, dt$

$= \frac{1}{2}\int_0^{2\pi} 190\,dt = \frac{1}{2}\times 2\times\pi\times 190 = 190\pi$

B) If $x(t) = 5\cos^3(t)$, then $x^{2/3} + y^{2/3} = 5^{2/3}$

so $5^{2/3}\cos^2 t + y^{2/3} = 5^{2/3}$ so $y^{2/3} = 5^{2/3}(1-\cos^2 t) = 5^{2/3}\sin^2 t$

so $y(t) = 5\sin^3 t$.

So a parametrization is given by

$\vec{r}(t) = \langle 5\cos^3 t, 5\sin^3 t\rangle. \quad 0 \le t \le 2\pi$

$\vec{v}(t) = \langle -1.5\sin t\cos^2 t, 1.5\cos t\sin^2 t\rangle$

$\text{area} = \frac{1}{2}\oint_C \langle -y, x\rangle \cdot d\vec{r} = \frac{1}{2}\int_0^{2\pi} \langle -5\sin^3 t, 5\cos^3 t\rangle \cdot$

$\langle -1.5\sin t\cos^2 t, 1.5\cos t\sin^2 t\rangle\, dt = \frac{1}{2}\int_0^{2\pi} (7.5\sin^4 t\cos^2 t$

$+ 7.5\cos^4 t\sin^2 t)\, dt$

$$= \frac{1}{2}\int_0^{2\pi} 75\,(\cos^2 t\,\sin^2 t)(\cos^2 t + \sin^2 t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} 75\,\cos^2 t\,\sin^2 t\,dt = \frac{1}{2}\int_0^{2\pi} 75\,\frac{\sin^2(2t)}{4}\,dt$$

$$= \frac{75}{2}\int_0^{2\pi} \frac{1-\cos 4t}{8}\,dt = \frac{75}{2}\left(\frac{1}{8}t + \frac{\sin 4t}{32}\Big|_0^{2\pi}\right)$$

$(\cos 4t = 1 - 2\sin^2(2t))$

$$= \frac{75}{2} \times \frac{1}{8} \times 2\pi = \frac{75}{8}\pi$$

4. $\vec{F} = \langle \underset{M}{\underset{\shortparallel}{\cos z}}, \underset{N}{\underset{\shortparallel}{2y^{17}}}, \underset{P}{\underset{\shortparallel}{-x\sin z}} \rangle$

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x} \qquad \frac{\partial M}{\partial z} = -\sin z = \frac{\partial P}{\partial x} \qquad \frac{\partial N}{\partial z} = 0 = \frac{\partial P}{\partial y}.$$

So $\vec{F}$ is conservative (and since it is defined everywhere, and the whole space is connected) it is a gradient field.

$\vec{F} = \nabla f$

You can guess what the general form of looks like by looking at $\vec{F}$ or you can take integrals:

$\frac{\partial f}{\partial x} = \cos z$, so $f(x,y,z) = \int \cos z\, dx$

$$f(x,y,z) = x\cos z + g(y,z) \qquad (*)$$

since we know $\frac{\partial f}{\partial y} = 2y^{17}$, we get

$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial y} - \frac{\partial}{\partial y}(x\cos z) = 2y^{17}$$

so $g(y,z) = \int 2y^{17}\, dy = \frac{1}{9} y^{18} + h(z)$

so $f(x,y,z) = x\cos(z) + \frac{1}{9} y^{18} + h(z)$ $\qquad (**)$

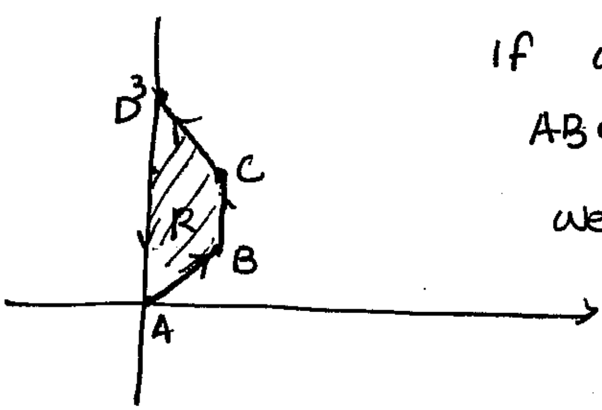
since we know $\frac{\partial f}{\partial z} = -x\sin(z)$, from the above line, we get

$$\frac{dh}{dz} = \frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\left(x\cos z + \frac{1}{9} y^{18}\right) = -x\sin(z) - x\cos z = 0$$

$\Rightarrow$ h is a constant function c, so

$$f(x,y,z) = x\cos z + \frac{1}{9} y^{18} + c \qquad (***)$$

5. You can either directly parametrize the three pieces and compute the line integral, or you can use Green's theorem in the following way:



If we look at the close curve $=$ ABCDA, Using Green's Theorem we get:

$\vec{F} = \langle \sin x + 4y,\ 12x+y \rangle$

$$\int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} + \int_{DA} \vec{F}\cdot d\vec{r}$$

$$= \iint_R \frac{\partial}{\partial x}(12x+y) - \frac{\partial}{\partial y}(\sin x + 4y)\ dA = \iint_R 12-4\ dA$$

$$= 8 \iint_R 1\ dA = 8 \text{ area of } R = 8\cdot\left(\tfrac{1}{2}+1+\tfrac{1}{2}\right) = 16$$

So $\int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CD} \vec{F}\cdot d\vec{r} = 16 - \int_{DA} \vec{F}\cdot d\vec{r} = 16 + \int_{AD} \vec{F}\cdot d\vec{r}$ (1)

And $\overline{AD}$ is parametrized by $\vec{r}(t) = \langle 0, t\rangle \quad 0 \le t \le 3$

so $\vec{v}(t) = \langle 0, 1\rangle$, so

$$\int_{AD} \vec{F}\cdot d\vec{r} = \int_0^3 \langle \sin(0) + 4t,\ 12\times 0 + t\rangle \cdot \langle 0,1\rangle\, dt$$

$$= \int_0^3 t\ dt = \frac{t^2}{2}\Big|_0^3 = \frac{9}{2} \quad (2)$$

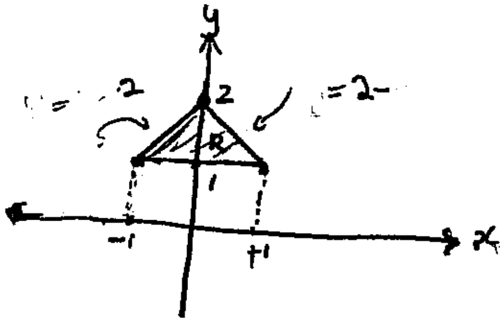
So $\int_{ABCD} (\sin x + 4y)\,dx + (12x+y)\,dy = 16 + \frac{9}{2} = 20.5$

by (1) and (2)

6. the surface is parametrized by

$$\vec{r}(x,y) = \langle x, y, y^3\cos^5(x)\rangle$$

(x,y) belongs to the triangle



so $\quad +1 \le y \le 2 \quad\quad y-2 \le x \le 2-y$

$$\vec{r}_x = \langle 1, 0, -5\sin(x)\cos^4(x)\, y^3 \rangle$$

$$\vec{r}_y = \langle 0, 1, 3y^2 \cos^5(x) \rangle$$

$$\vec{r}_x \times \vec{r}_y = \langle +5\sin(x)\cos^4(x)\, y^3, -3y^2\cos^5(x), 1 \rangle$$

$$\Rightarrow |\vec{r}_x \times \vec{r}_y| = \sqrt{\underbrace{1 + 25\sin^2(x)\cos^8(x)\, y^6 + 9y^4\cos^{10}(x)}_{h(x,y)}}$$

surface area $= \iint_R \sqrt{h(x,y)}\, dA = \int_1^2 \int_{y-2}^{-y+2} \sqrt{h(x,y)}\, dA$

7. $\quad r(\theta, \phi) = \langle 2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi \rangle$

$$\vec{r}_\theta = \langle -2\sin\theta\sin\phi, 2\cos\theta\sin\phi, 0 \rangle$$

$$\vec{r}_\phi = \langle 2\cos\theta\cos\phi, 2\sin\theta\cos\phi, -2\sin\phi \rangle$$

when $\theta = \frac{\pi}{4}$, $\phi = \frac{\pi}{4}$

$\vec{r}_\theta = \langle -1, 1, 0 \rangle$ $\quad\quad$ so $\quad \vec{r}_\theta \times \vec{r}_\phi = \langle -\sqrt{2}, -\sqrt{2}, \cancel{0}^{\,-2} \rangle$

$\vec{r}_\phi = \langle +1, 1, -\sqrt{2} \rangle$

$f(x,y,z) = x^2+y^2+z^2$

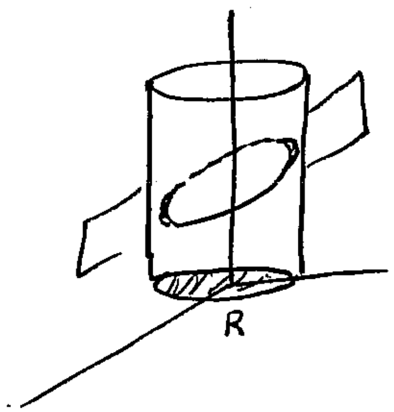
$\nabla f = \langle 2x, 2y, 2z \rangle$

at $\phi = \frac{\pi}{4}$ $\theta = \frac{\pi}{4}$ $\vec{r}(\theta, \phi) = \langle 1, 1, \sqrt{2} \rangle$

so ∇f at $(1, 1, \sqrt{2}) = \langle 2, 2, 2\sqrt{2} \rangle$

they are parallel because they are both normal to the sphere (i.e. they are orthogonal to the tangent plane of the sphere).

8.



The points on the plane are parametrized by

$$\vec{r}(x,y) = \langle x, y, 8-8x-10y \rangle$$

If we look at the points which are inside the elliptic cylinder (this means that the base of the cylinder is an ellipse instead of a circle). then we are saying that our parameters come from the ~~base or~~ region R enclosed by the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

$\vec{r}_x = \langle 1, 0, -8 \rangle$ $\quad \vec{r}_x \times \vec{r}_y = \langle 8, 10, 1 \rangle$ $\quad |\vec{r}_x \times \vec{r}_y| = \sqrt{165}$

$\vec{r}_y = \langle 0, 1, -10 \rangle$

therefore surface area of the part of the plane in the question S is

$$\text{area} = \iint_S 1 \, d\sigma = \iint_R |\vec{r}_x \times \vec{r}_y| \, dA = \iint_R \sqrt{165} \, dA = \sqrt{165} \iint_R 1 \, dA$$

$$= \sqrt{165} \text{ area of } R = \sqrt{165} \times \underbrace{\pi \times 4 \times 3} = 12\sqrt{165}\,\pi$$

We showed in class Using Green's Theorem that the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is πab.

9. (a) $\vec{T}_u = \vec{r}_u = \langle 3, 1, 5 \rangle$ $\quad \vec{r}_u \times \vec{r}_v = \langle 6, -3, -3 \rangle$

$\vec{T}_v = \vec{r}_v = \langle 0, -1, 1 \rangle$ $\quad |\vec{r}_u \times \vec{r}_v| = \sqrt{36+9+9} = \sqrt{54}$

(b) area of $S = \iint_S 1 \, d\sigma = \iint_D \sqrt{54} \, dA$ where

D is $= (u,v) : 0 \le u \le 8, \ 0 \le v \le 7$

So area $= \int_0^8 \int_0^7 \sqrt{54} \, dv \, du = \sqrt{54} \cdot 7 \cdot 8 = 56\sqrt{54}$

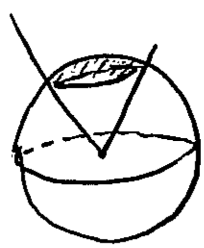
(c). $f(x,y,z) = yz = (u-v)(5u+v)$

$$\iint_S f\, d\sigma = \int_0^8 \int_0^7 \sqrt{54}\ (u-v)(5u+v)\ dv\, du$$

$$= \sqrt{54} \int_0^8 \int_0^7 5u^2 - v^2 - 4uv\ dv\, du = \sqrt{54} \int_0^8 5u^2 v - \frac{v^3}{3} - 2uv^2 \Big|_0^7 du$$

$$= \sqrt{54} \int_0^8 35u^2 - \frac{7^3}{3} - 98u\ du = \sqrt{54}\left(\frac{35}{3}u^3 - \frac{7^3}{3}u - 49u^2 \Big|_0^8\right)$$

$$= \sqrt{54}\left(\frac{35 \times 8^3}{3} - \frac{7^3 \times 8}{3} - 49 \times 8^2\right)$$

10. 

The cone and the sphere meet along the curve $z = \sqrt{x^2+y^2}$, $x^2+y^2+z^2 = 100$, so $2(x^2+y^2) = 100$, so $\underline{x^2+y^2 = 50}$ and $\underline{z = \sqrt{50}}$

<u>solution 1</u>: we use x, y as parameters: we get a

parametrization of the part of the sphere above the cone $\quad \vec{r}(x,y) = \langle x, y, \sqrt{100-x^2-y^2} \rangle$

(x,y) belongs to the disk of radius $\sqrt{50}$ around the origion on the xy-plane. call it R.

$$\vec{r}_x = \left\langle 1, 0, \frac{-x}{\sqrt{100-x^2-y^2}} \right\rangle$$

$$\vec{r}_y = \left\langle 0, 1, \frac{-y}{\sqrt{100-x^2-y^2}} \right\rangle$$

$\vec{r}_x \times \vec{r}_y = \langle \frac{x}{}, \frac{y}{}, 1 \rangle$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{\frac{x^2}{100-x^2-y^2} + \frac{y^2}{100-x^2-y^2} + 1} = \frac{10}{\sqrt{100-x^2-y^2}}$$

so surface area

$$= \iint_R \frac{10}{\sqrt{100-x^2-y^2}}\, dA \ \overset{\text{polar}}{\underset{\text{coordinates}}{=}} \ \int_0^{2\pi}\int_0^{\sqrt{50}} \frac{10}{\sqrt{100-r^2}}\, r\, dr$$

$$= \int_0^{2\pi}\int_0^{\sqrt{50}} 10r\,(100-r^2)^{-\frac{1}{2}}\, dr\, d\theta = \int_0^{2\pi} -10\,(100-r^2)^{\frac{1}{2}}\Big|_0^{\sqrt{50}}\, d\theta$$

$$= \int_0^{2\pi} -10\times\sqrt{50} + 10\times 10\ d\theta = (100 - 10\sqrt{50})\, 2\pi$$

Solution 2: you can also use spherical coordinates:

points on the sphere of radius 10 are parametrized by

$$\vec{r}(\theta,\phi) = \langle 10\sin\phi\cos\theta,\ 10\sin\phi\sin\theta,\ 10\cos\phi\rangle$$

$$|\vec{r}_\theta \times \vec{r}_\phi| = 100\sin\phi$$ (look at page 891 of the book)

the equation of the cone in spherical coordinates is given by

$$\underbrace{\rho\cos\phi}_{z} = \sqrt{\underbrace{\rho^2\sin^2\phi\cos^2\theta}_{x^2} + \underbrace{\rho^2\sin^2\phi\sin^2\theta}_{y^2}} = \rho\sin\phi$$

so $\cos\phi = \sin\phi$ so $\phi = \frac{\pi}{4}$.

Therefore, the bounds for S are given ~~to~~ by
$0 \le \theta \le 2\pi \quad 0 \le \phi \le \frac{\pi}{4}$, so we get

$$\text{surface area} = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} 100 \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} -100 \cos\phi \Big|_0^{\frac{\pi}{2}} d\theta$$

$$= \int_0^{2\pi} 100 \left(1 - \frac{\sqrt{2}}{2}\right) d\theta = (100 - 50\sqrt{2})\, 2\pi = (100 - 10\sqrt{50})\, 2\pi.$$

11). We use the ~~parametrization~~ parameters ~~x~~ and z.

$$\vec{r}(x,z) = \langle x, 4-z^2, z \rangle \qquad \begin{array}{l} 0 \le x \le 6 \\ 0 \le z \le 6 \end{array}$$

$$\vec{r}_x = \langle 1, 0, 0 \rangle \qquad \vec{r}_x \times \vec{r}_z = \langle 0, -1, -2z \rangle$$

$$\vec{r}_z = \langle 0, -2z, 1 \rangle \qquad |\vec{r}_x \times \vec{r}_z| = \sqrt{1+4z^2}$$

$$\iint_S F \, d\sigma = \int_0^6 \int_0^6 z \sqrt{1+4z^2} \, dz \, dx = \int_0^6 \frac{1}{12} (1+4z^2)^{\frac{3}{2}} \Big|_{z=0}^{6} dx$$

$$= \int_0^6 \frac{1}{12} \left(145^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) dx = \frac{1}{2} \left(145^{\frac{3}{2}} - 1\right)$$