

1. If  $\mathbf{r}_1(t) = \langle t^2, e^t, t+2 \rangle$ , and  $\mathbf{r}_2(t)$  is a vector function such that

$$\mathbf{r}_2(0) = \langle 1, 1, -1 \rangle, \quad \mathbf{r}'_2(0) = \langle 3, 1, 0 \rangle,$$

what is  $(\mathbf{r}_1 \cdot \mathbf{r}_2)'(0)$ ?

(a) -2

(b) -1

(c) 0

(d) 1

(e) 2

(f) 3

$$\vec{\mathbf{r}}_1(0) = \langle 0, 1, 0 \rangle \quad \vec{\mathbf{r}}'_1(t) = \langle 2t, e^t, 1 \rangle, \text{ so } \vec{\mathbf{r}}'_1(0) = \langle 0, 1, 1 \rangle$$

$$\begin{aligned} \frac{d}{dt} (\vec{\mathbf{r}}_1 \cdot \vec{\mathbf{r}}_2)(0) &= \vec{\mathbf{r}}_1(0) \cdot \vec{\mathbf{r}}'_2(0) + \vec{\mathbf{r}}'_1(0) \cdot \vec{\mathbf{r}}_2(0) \\ &= \langle 0, 1, 0 \rangle \cdot \langle 3, 1, 0 \rangle + \langle 0, 1, 1 \rangle \cdot \langle 1, 1, -1 \rangle \\ &= 1 + 0 \\ &= 1 \end{aligned}$$

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2. What is the parametric equation of the tangent line to the curve traced by  $\mathbf{r}(t) = \langle t, \frac{1}{1+t}, t^2 \rangle$  at  $t = 0$ ?

(a)  $x = 1 + t, y = 2 + t, z = 1 + t$

(b)  $x = t, y = 1 - t, z = t$

(c)  $x = 2t, y = 1 + t, z = t$

(d)  $x = 1 + t, y = 1 - t, z = 1$

(e)  $x = t, y = 1 + t, z = 0$

(f)  $x = t, y = 1 - t, z = 0$

$$\vec{r}'(t) = \left\langle 1, \frac{-1}{(1+t)^2}, 2t \right\rangle$$

$$\vec{r}'(0) = \langle 1, -1, 0 \rangle \quad \vec{r}(0) = \langle 0, 1, 0 \rangle$$

L: the tangent line

L is parallel to  $\langle 1, -1, 0 \rangle$  and passes through  $(0, 1, 0)$

so

$$L: x = t \quad y = 1 - t \quad z = 0$$

3. If  $\mathbf{r}(t)$  is a vector function such that

$$\mathbf{r}''(t) = -2tk$$

and

$$\mathbf{r}(0) = 3k, \quad \mathbf{r}'(0) = i + j,$$

then what is  $\mathbf{r}(3)$ ?

(a)  $<1, 3, 6>$

(b)  $<1, 3, 3>$

$\textcircled{(c)} <3, 3, -6>$

(d)  $<3, 3, 6>$

(e)  $<1, 1, 3>$

(f)  $<1, -1, 3>$

$$\vec{r}(t) = \int \vec{r}''(t) + \vec{C} = \langle 0, 0, \frac{-t^2}{2} \rangle + \vec{C}$$

$\vec{C} = \langle 1, 1, 0 \rangle$

since  $\vec{r}(0) = \langle 1, 1, 0 \rangle$ , we have

$$\vec{r}'(t) = \langle 1, 1, \frac{-t^2}{2} \rangle$$

$$\vec{r}(t) = \int \vec{r}'(t) + \vec{C} = \langle 1, 1, -\frac{t^3}{6} \rangle + \vec{C}$$

since  $\vec{r}(0) = \langle 0, 0, 3 \rangle$ , we have  $\vec{C} = \langle 0, 0, 3 \rangle$ ,

$$\vec{r}(t) = \langle 1, 1, -\frac{t^3}{6} + 3 \rangle, \text{ so } \vec{r}(3) = \langle 3, 3, -6 \rangle$$

4. Find the following limit

$$\lim_{(x,y) \rightarrow (6,3)} \frac{x^2 - xy - 2y^2}{x - 2y}.$$

(a) 0

(b) 1

(c) 3

(d) 6

(e) 9

(f) The limit does not exist.

$$\frac{x^2 - xy - 2y^2}{x - 2y} = \frac{(x-2y)(x+y)}{x-2y}$$

$$\text{so } \lim_{(x,y) \rightarrow (6,3)} \frac{(x-2y)(x+y)}{x-2y} = \lim_{(x,y) \rightarrow (6,3)} x+y = 9$$

5. If  $f(x, y) = x^2 \sin(y^2)$ , then what is  $f_{xyx}$ ?

(a) 0

(b)  $2 \sin(y^2)$

(c)  $2 \cos(y^2)$

(d)  $4y \cos(y^2)$

(e)  $2y \cos(y^2)$

(f)  $-2y \cos(y^2)$

6. What is the linearization of the function  $f(x, y) = \frac{1}{x^2-y}$  at  $(-1, 0)$ ?

(a)  $3 + 2x + y$

(b)  $-1 + 2x + y$

(c)  $-1 - 2x + y$

(d)  $3 + 2x - y$

(e)  $-1 + 2x - y$

(f)  $-1 - 2x - y$

$$f(-1, 0) = 1$$

$$\frac{\partial f}{\partial x} = \frac{-2x}{(x^2-y)^2}, \text{ so } \frac{\partial f}{\partial x}(-1, 0) = 2$$

$$\frac{\partial f}{\partial y} = \frac{1}{(x^2-y)^2}, \text{ so } \frac{\partial f}{\partial y}(-1, 0) = 1$$

so  $L(x, y) = 1 + 2(x - (-1)) + (y - 0)$   
 $= 3 + 2x + y$

7. Suppose that  $T$  is given by  $x(e^y + e^{-y})$  where  $x$  and  $y$  are found to be 2 and  $\ln 2$  with a possible error of 0.2 in  $x$  and 0.1 in  $y$ . Using differentials, estimate the maximum possible error in the computed value of  $T$ .

(a) 0.1

(b) 0.2

(c) 0.4

(d) 0.8

(e) 1

(f) 2

$$dT = \frac{\partial T}{\partial x} dx + \frac{\partial T}{\partial y} dy$$

$$dx = 0.2 \quad dy = 0.1 \quad \frac{\partial T}{\partial x} = e^y + e^{-y} \quad \frac{\partial T}{\partial y} \text{ (cancel)}$$

$$\frac{\partial T}{\partial y} = x(e^y - e^{-y})$$

$$\text{So } dt = 0.2 (e^y + e^{-y}) + 0.1 \times 2 \times (e^y - e^{-y})$$

$$= 0.2 (e^y + e^{-y} + e^y - e^{-y}) = 0.4 e^y = 0.4 \times 2 = 0.8$$

$$y = \ln 2$$

8. The set of points in the plane at which the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2 - y^2} & \text{if } x \neq \pm y \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is not continuous is:

(a) A line

(b) A line with a point removed

(c) a point

(d) the function is continuous everywhere.

(e) two intersecting lines

(f) two intersecting lines with their intersection removed

The function is not defined when  $x = \pm y$  and  $(x, y) \neq (0, 0)$

At the origin, we use the 2-path test to show the limit does not exist.

On the  $x$ -axis  $f(x, y) = 0$

On the line  $x = 2y$   $f(x, y) = \frac{2y}{4y^2 - y^2} = \frac{2}{3}$

9. What is the largest possible  $z$ -coordinate a point on the curve of intersection of the surfaces  $x^2 + 4y^2 = 1$  and  $z - 2y - x = 0$  can have?

(a) 1

(b)  $\sqrt{2}$

(c) 2

(d)  $\sqrt{3}$

(e) 3

(f) 4

We first parametrize the curve:

$$x^2 + 4y^2 = 1 : \quad x = \sin t \quad 2y = \cos t, \text{ so } y = \frac{\cos t}{2}$$

$$z - 2y - x = 0 : \quad z = 2y + x = 2 \cdot \frac{\cos t}{2} + \sin t = \cos t + \sin t$$

so we need to find the maximum of  $\cos t + \sin t$ .

The derivative is  $-\sin t + \cos t$  which is zero when

$$\sin t = \cos t, \text{ so } t = k\pi + \frac{\pi}{4}, \text{ so } \cos t + \sin t = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$$

10. The level curve of  $f(x, y) = \frac{2x^2+2+2y}{x^2+y^2+1}$  which passes through  $(1, 0)$  is

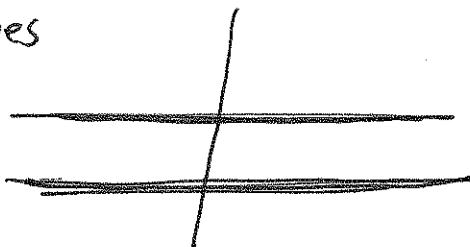
- (a) a point
- (b) a circle of radius 2
- (c) two parallel lines
- (d) two intersecting lines
- (e) an ellipse
- (f) a circle of radius 1

$$f(1, 0) = \frac{2+2}{1+1} = 2.$$

If  $f(x, y) = 2$ , then  $\frac{2x^2+2+2y}{x^2+y^2+1} = 2$ , so

$$2x^2 + 2 + 2y = 2x^2 + 2y^2 + 2, \text{ so } y = y^2 \text{ so } y = 1, \text{ or } y = 0$$

so we get two parallel lines



11. A rectangular box has sides whose lengths  $a$ ,  $b$ , and  $c$  are changing with time. At an instant  $a = 1\text{m}$ ,  $b = 1\text{m}$ ,  $c = 2\text{m}$ , and  $\frac{da}{dt} = 2 \text{ m/s}$ ,  $\frac{db}{dt} = 1 \text{ m/s}$ ,  $\frac{dc}{dt} = -3 \text{ m/s}$ . What is the rate at which the length of the big diagonal of the box is changing?

(a)  $\sqrt{2}$

(b)  $-\sqrt{2}$

(c)  $\sqrt{6}$

(d)  $-\sqrt{6}$

(e)  $\sqrt{\frac{3}{2}}$

(f)  $-\sqrt{\frac{3}{2}}$

$$d = \sqrt{x^2 + y^2 + z^2}$$

$$f_x = \frac{x}{\sqrt{x^2 + y^2 + z^2}}$$

$$f_x(1, 1, 2) = \frac{1}{\sqrt{6}}$$

$$f_y = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

$$f_z = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$f_y(1, 1, 2) = \frac{1}{\sqrt{6}}$$

$$f_z(1, 1, 2) = \frac{2}{\sqrt{6}}$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} = \frac{-2}{\sqrt{6}} + \frac{1}{\sqrt{6}} + \frac{-3}{\sqrt{6}} = \frac{-4}{\sqrt{6}}$$

12. Suppose that  $f(x, y)$  is such that  $f_x = x^2 + y + 2xy$ ,  $f_y = y^2 + x^2 + x$ , and  $f(1, 1) = 4$ . What is a vector parallel to the line tangent to the intersection of the graph of  $f$  and the plane  $x = 1$  at  $(1, 1, 4)$ ?

(a)  $\langle 4, 3, 1 \rangle$

(b)  $\langle 1, 0, 4 \rangle$

(c)  $\langle 1, 0, 3 \rangle$

(d)  $\langle 0, 1, 3 \rangle$

(e)  $\langle 0, 1, 4 \rangle$

(f)  $\langle 3, 4, 1 \rangle$

A vector parallel to the tangent line is

$$\langle 0, 1, \frac{\partial f}{\partial y}(1, 1) \rangle$$

$$\frac{\partial f}{\partial y}(1, 1) = f_y(1, 1) = 1+1+1=3$$

13. In what direction is the rate of change of  $f(x, y) = xy + y^2$  at  $(-1, 1)$  equal to 0?

(a)  $\langle 1, 0 \rangle$

(b)  $\langle 0, 1 \rangle$

(c)  $\langle \frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}} \rangle$

(d)  $\langle \frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}} \rangle$

(e)  $\langle \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \rangle$

(f)  $\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \rangle$

$$f_x = y \quad f_y = x + 2y$$

$$\nabla f(-1, 1) = \langle 1, 1 \rangle$$

$$\text{If } D_{\vec{u}} f(-1, 1) = \vec{u} \cdot \langle 1, 1 \rangle$$

If  $\vec{u} = \langle a, b \rangle$ , we have  $\langle a, b \rangle \cdot \langle 1, 1 \rangle = 0$

$$\text{so } a+b=0 \quad \text{so } a=-b$$

14. The direction of  $f(x, y, z)$  at a point  $P$  is greatest in the direction  $i + j - k$ , and the value of the derivative in this direction is  $2\sqrt{3}$ . What is  $f_x$  at that point?

(a) 1

(b) -1

(c) 2

(d) -2

(e) 4

(f) -4

a Directional derivative is maximum in the direction of

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

Since it is parallel to  $\langle 1, 1, -1 \rangle$ , we have

$$\langle f_x, f_y, f_z \rangle = c \langle 1, 1, -1 \rangle$$

$$\text{so } |\nabla f| = c\sqrt{3} \quad \text{since } |\nabla f| = 2\sqrt{3}, \quad c = 2, \text{ so } f_x = 2$$

15. What is the minimum distance from the point  $(1, 1, 1)$  to the surface  $x^2 - 2x - yz = 0$ ?

(a)  $\sqrt{\frac{5}{3}}$

(b)  $\sqrt{3}$

(c)  $\sqrt{5}$

(d)  $\frac{3}{5}$

(e)  $\frac{1}{3}$

(f)  $\frac{1}{5}$

We want to minimize

$$d^2 = (x-1)^2 + (y-1)^2 + (z-1)^2 = x^2 + y^2 + z^2 - 2x - 2y - 2z + 3$$

But  $x^2 - 2x - yz = 0$ , so

$$d^2 = \underbrace{yz + y^2 + z^2 - 2y - 2z + 3}_{f(y, z)}$$

$$f_y = z + 2y - 2 \quad f_z = y + 2z - 2$$

$$\begin{cases} z + 2y = 2 \\ y + 2z = 2 \end{cases} \Rightarrow z = y = \frac{2}{3} \quad \text{so } d^2 = \frac{4}{9} + \frac{4}{9} + \frac{4}{9} - \frac{4}{3} - \frac{4}{3} + 3 = \frac{5}{9}$$

16. How many local maximum and minimum points does the following function have?

$$f(x, y) = 4xy - x^4 - y^4$$

(a) 1 local maximum and 1 local minimum

(b) 2 local maximum and no local minimum

(c) 1 local maximum and no local minimum

(d) 2 local maximum and 2 local minimum

(e) no local maximum and 2 local minimum

(f) 1 local maximum and 2 local minimum

$$f_x = 4y - 4x^3 \quad f_y = 4x - 4y^3$$

critical points :  $\begin{cases} 4y = 4x^3 & , \text{ so } y = x^3 \\ 4x = 4y^3 & \text{ so } x = y^3 = (x^3)^3 = x^9 \end{cases}$

Since  $x = x^9$  either  $x=0$  or  $x^8=1$ , so  $x=\pm 1$

$$f_{xx} = -12x^2 \quad f_{xy} = 4 \quad f_{yy} = -12y^2 \quad \begin{array}{l} \text{if } x=1, y=1 \\ \text{if } x=-1, y=-1 \end{array}$$

at  $(0,0)$   $D = \det \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$ , so  $D < 0$  saddle point  
 at  $(1,1)$   $D = \det \begin{pmatrix} -12 & 4 \\ 4 & -12 \end{pmatrix} > 0$   $f_{xx} > 0$  so we have a local maximum  
 and  $(-1,-1)$