

1. Evaluate

$$\int_0^3 \int_0^4 x - \sqrt{y} \ dy \ dx$$

(a)  $-\frac{5}{6}$

(b)  $\frac{1}{2}$

(c) 1

(d)  $\frac{7}{6}$

(e) 2

(f) 3

$$\begin{aligned} \int_0^3 \int_0^4 x - \sqrt{y} \ dy \ dx &= \int_0^3 \left[ xy - \frac{2}{3} y^{\frac{3}{2}} \right]_{y=0}^{y=4} dx \\ &= \int_0^3 \left[ 4x - \frac{16}{3} \right] dx = \left[ 2x^2 - \frac{16}{3} x \right]_0^3 \\ &= 18 - 16 = 2 \end{aligned}$$

2

2. Estimate the double integral of  $f(x, y) = \frac{2x}{y+1}$  over the rectangular region  $[0, 2] \times [0, 4]$  using  $n = m = 2$  and the midpoints.

(a) 3

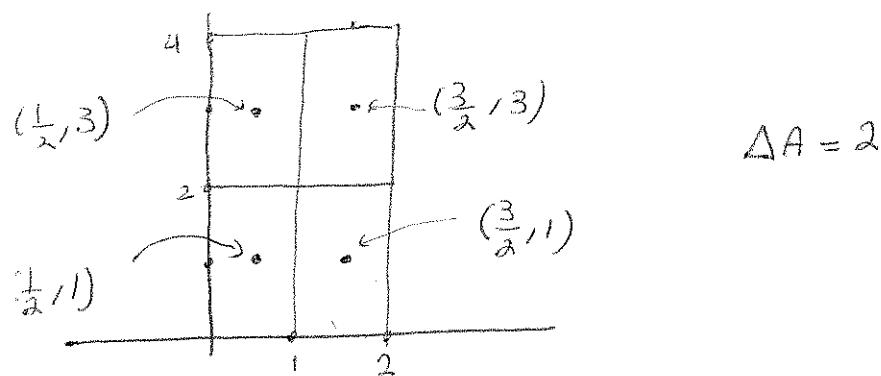
(b) 3.5

(c) 4

(d) 5.5

(e) 6

(f) 6.5



$$\begin{aligned} \iint_R f(x,y) dA &\approx \left( f\left(\frac{1}{2}, 1\right) + f\left(\frac{3}{2}, 1\right) + f\left(\frac{1}{2}, 3\right) + f\left(\frac{3}{2}, 3\right) \right) \Delta A \\ &= \left( \frac{1}{2} + \frac{3}{2} + \frac{1}{4} + \frac{3}{4} \right) 2 = 6 \end{aligned}$$

3. Find the double integral of

$$f(x, y) = 3x$$

over the region in the first quadrant bounded by the lines  $y = x$ ,  $y = -x + 4$ ,  $x = 1$ .

(a) 3

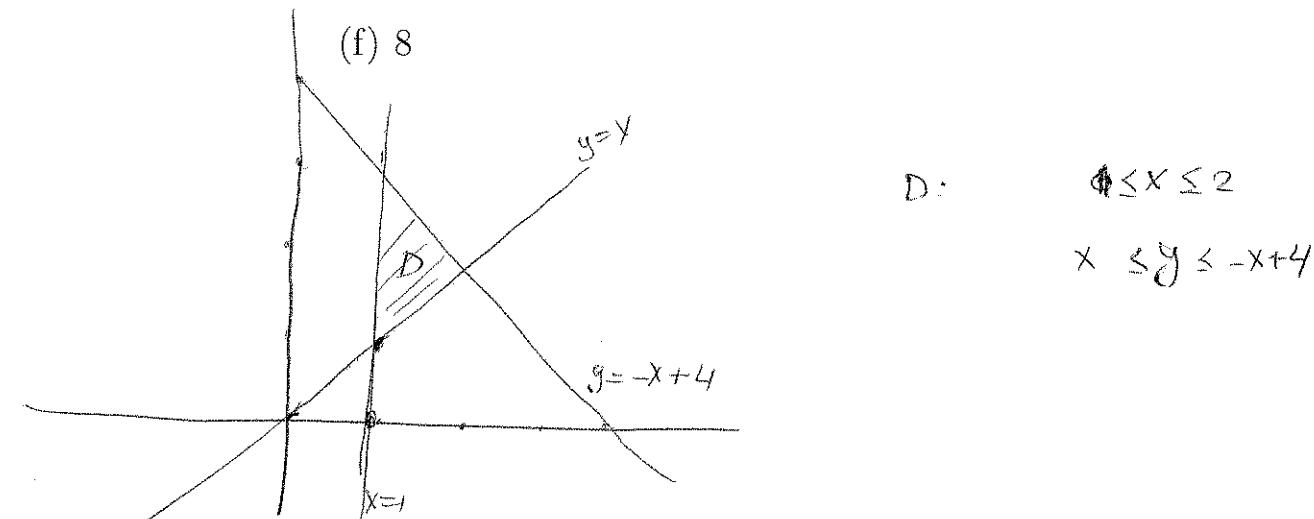
(b) 4

(c) 5

(d) 6

(e) 7

(f) 8



$$\iint_D 3x \, dA = \int_1^2 \int_x^{-x+4} 3x \, dy \, dx = \int_1^2 3x y \Big|_{y=x}^{y=-x+4} \, dx$$

$$= \int_1^2 3x(-x+4) - 3x^2 \, dx = \int_1^2 (12x - 6x^2) \, dx = 6x^2 - 2x^3 \Big|_1^2 = (24 - 16) - (6 - 2) = 4$$

4. Find the average value of  $f(x, y) = \sin(x + y)$  over the rectangular region  $0 \leq x \leq \pi, 0 \leq y \leq \pi$ .

(a) 0

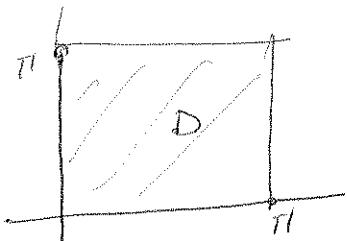
(b) 1

(c) 2

(d)  $\frac{1}{\pi^2}$

(e)  $\frac{2}{\pi^2}$

(f)  $\frac{4}{\pi^2}$



$$\text{area of } D = \pi^2$$

$$\begin{aligned}
 \iint_D \sin(x+y) \, dA &= \int_0^\pi \int_0^\pi \sin(x+y) \, dx \, dy = \int_0^\pi -\cos(x+y) \Big|_{x=0}^{x=\pi} \, dy \\
 &= \int_0^\pi -\cos(\pi+y) + \cos y \, dy = -\sin(\pi+y) + \sin y \Big|_0^\pi \\
 &= -\sin 2\pi + \sin \pi + \sin \pi - \sin 0 = 0
 \end{aligned}$$

5. Evaluate the following integral

$$\int_0^1 \int_y^1 x^2 e^{xy} dx dy.$$

(a)  $\frac{e}{2}$

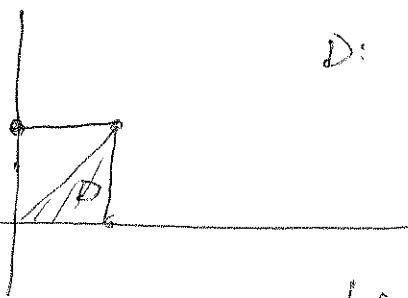
(b)  $\frac{e}{2} - 1$

(c)  $e$

(d)  $\frac{e-1}{2}$

(e) 1

(f)  $e - 1$



$D: 0 \leq y \leq 1$   
 $y \leq x \leq 1$

$D: 0 \leq x \leq 1$   
 $0 \leq y \leq x$

reversing the int order:

$$\iint_D x^2 e^{xy} dA = \int_0^1 \int_0^x x^2 e^{xy} dy dx$$

$$= \int_0^1 x e^{xy} \Big|_0^x dx = \int_0^1 x e^x - x dx = \left[ \frac{e^x}{2} - \frac{x^2}{2} \right]_0^1$$

$$= \frac{e-1}{2} - \left( \frac{1-0}{2} \right) = \frac{e-1}{2} - 1$$

6. Find the area of the region bounded by the parabolas  $x = y^2 - 1$  and  $x = 2y^2 - 2$ .

(a)  $\frac{1}{2}$

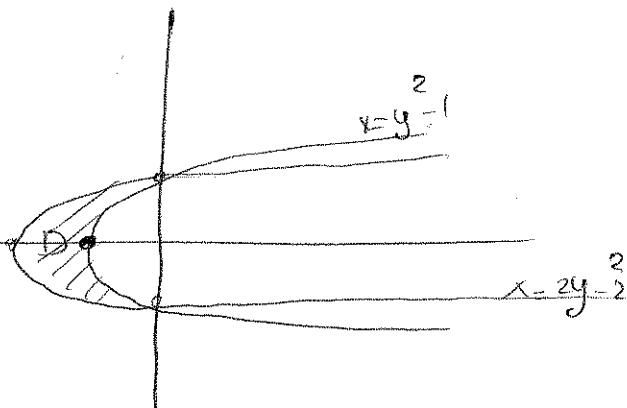
(b)  $\frac{2}{3}$

(c)  $\frac{4}{3}$

(d)  $\sqrt{3}$

(e) 2

(f) 3



$$D: \begin{cases} -1 \leq y \leq 1 \\ 2y^2 - 2 \leq x \leq y^2 - 1 \end{cases}$$

$$\begin{aligned} \text{area of } D &= \iint_D 1 \, dA = \int_{-1}^1 \int_{2y^2-2}^{y^2-1} 1 \, dx \, dy \\ &= \int_{-1}^1 x \Big|_{2y^2-2}^{y^2-1} \, dy = \int_{-1}^1 (y^2 - 1) - (2y^2 - 2) \, dy \\ &= \int_{-1}^1 -y^2 + 1 \, dy = -\frac{y^3}{3} + y \Big|_{-1}^1 = \frac{4}{3} \end{aligned}$$

7. What is the curve whose equation in polar coordinates is

$$r = 2 \cos \theta + 2 \sin \theta?$$

(a) a circle of radius 1 around (2, 2)

(b) a circle of radius  $\sqrt{2}$  around (1, 1)

(c) a circle of radius  $\sqrt{2}$  around the origin

(d) two intersection circles

(e) two disjoint circles

(f) a circle of radius 1 around  $(\sqrt{2}, \sqrt{2})$

$$x = r \cos \theta \Rightarrow \cos \theta = \frac{x}{r}$$

$$y = r \sin \theta \Rightarrow \sin \theta = \frac{y}{r}$$

so the equation becomes:  $r = \frac{2x}{r} + \frac{2y}{r}$ ,  $r^2 = 2x + 2y$

so  $x^2 + y^2 = 2x + 2y$ , so  $(x-1)^2 + (y-1)^2 = 2$

8. Find the volume of the solid in the first octant enclosed by the surface  $z = 4 - y^2$  and the plane  $x = 3$ .

(a) 4

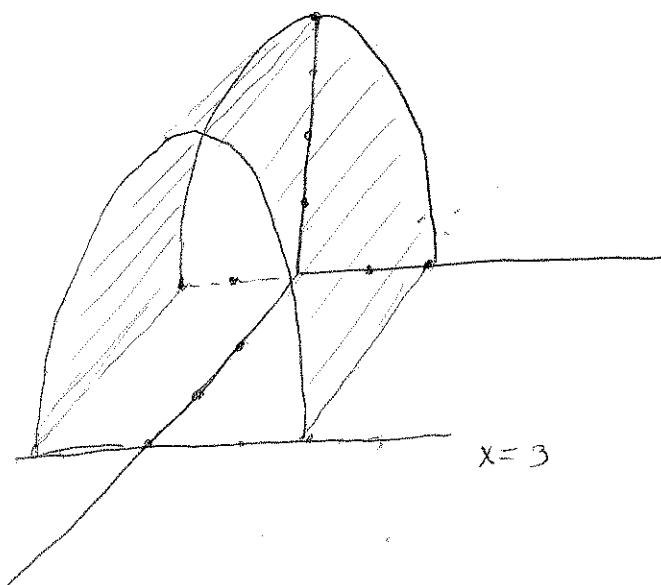
(b) 8

(c)  $\frac{14}{5}$

(d) 16

(e) 18

(f) 20



$$\begin{aligned}
 \text{Volume} &= \int_0^3 \int_0^2 4 - y^2 \, dy \, dx \\
 &= \int_0^3 \left[ 4y - \frac{y^3}{3} \right]_0^2 \, dx = \int_0^3 8 - \frac{8}{3} \, dx \\
 &= 3\left(8 - \frac{8}{3}\right) = 24 - 8 \\
 &= 16
 \end{aligned}$$

9. Use polar coordinates to evaluate

$$\int_0^1 \int_x^{\sqrt{2-x^2}} \sqrt{x^2 + y^2 + 1} dy dx.$$

(a)  $\frac{10\sqrt{10}\pi}{3}$

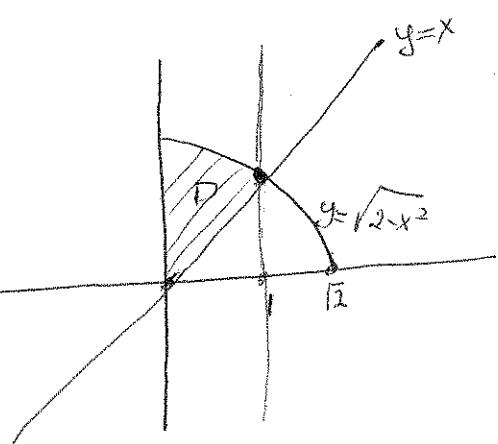
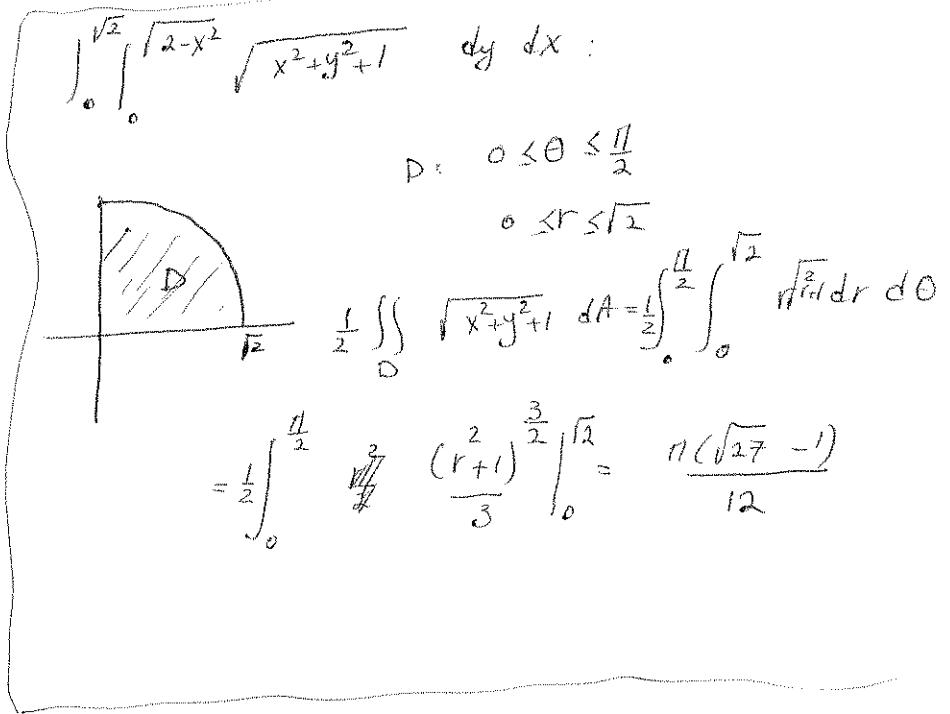
(b)  $\frac{10\sqrt{10}\pi}{6}$

(c)  $\frac{\pi(\sqrt{3}-\sqrt{2})}{6}$

(d)  $\frac{\pi(3\sqrt{3}-1)}{12}$

(e)  $\pi\sqrt{8}$

(f)  $\pi\sqrt{2}$



$y = \sqrt{2-x^2} \Rightarrow x^2 + y^2 = 2$  : a circle of radius 2

D:  $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$   
 $0 \leq r \leq \sqrt{2}$

$$\begin{aligned} \iint_D \sqrt{x^2 + y^2 + 1} dA &= \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} r \sqrt{r^2 + 1} dr d\theta = \int_{\pi/4}^{\pi/2} \left[ \frac{(r^2+1)^{3/2}}{3} \right]_0^{\sqrt{2}} d\theta \\ &= \int_{\pi/4}^{\pi/2} \frac{3\sqrt{27}-1}{3} d\theta = \left( \frac{\sqrt{27}-1}{3} \right) \frac{\pi}{4} = \frac{\pi(\sqrt{27}-1)}{12} \end{aligned}$$

10. What is the area of the region inside the curve  $r = 1 + \cos \theta$  and outside the circle of radius one around the origin?

(a)  $\pi$

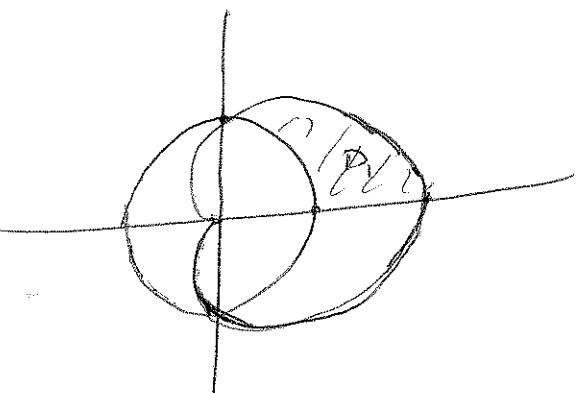
(b)  $\frac{\pi}{2}$

(c)  $\frac{\pi}{4}$

(d)  $\frac{\pi}{8}$

(e)  $1 + \frac{\pi}{8}$

(f)  $2 + \frac{\pi}{4}$



$$D_1: 0 \leq \theta \leq \frac{\pi}{2}$$

$$1 \leq r \leq 1 + \cos \theta$$

$$\text{area} = 2 \text{ area of } D_1 = 2 \int_0^{\frac{\pi}{2}} \int_1^{1+\cos \theta} r dr d\theta = 2 \int_0^{\frac{\pi}{2}} \frac{r^2}{2} \Big|_{1}^{1+\cos \theta} d\theta$$

$$= 2 \int_0^{\frac{\pi}{2}} \frac{(1+\cos \theta)^2 - 1}{2} d\theta = 2 \int_0^{\frac{\pi}{2}} \frac{\cos^2 \theta}{2} + \cos \theta d\theta = 2 \left( \sin \theta + \frac{\theta}{4} + \frac{\sin 2\theta}{8} \right) \Big|_0^{\frac{\pi}{2}}$$

$$= 2 \left( \frac{\pi}{8} + 1 \right) = \frac{\pi}{4} + 2$$

11. Find the largest product that the positive real numbers  $x, y, z$  can have if  $x^2 + y + z = 10$ .

(a)  $\frac{3}{4}$

(b) 18

(c)  $16\sqrt{2}$

(d)  $\frac{81}{4}$

(e)  $\frac{100\sqrt{10}}{9\sqrt{3}}$

(f) 20

maximize  $f(x, y, z) = xyz$  subject to  $\underbrace{x^2 + y + z}_{g(x, y, z)} = 10$

$$\begin{cases} yz = 2x\lambda \\ xz = \lambda \\ xy = \lambda \\ x^2 + y + z = 10 \end{cases} \Rightarrow y = z \Rightarrow \begin{aligned} y^2 &= 2x(xz) \Rightarrow y = 2x \\ z &= y = 2x \end{aligned}$$

so from  $x^2 + y + z = 10$ , we get  $\frac{y^2}{2} + y + y = 10$  so  $5y = 20$  so  $y = 4$ .

$z = 4, x^2 = 2$  so  $x = \sqrt{2}$ , so  $xyz = 16\sqrt{2}$

12. What is the equation of the tangent plane to the surface

$$2z = x^2$$

at  $(2, 0, 2)$ ?

(a)  $-2x + z = -2$

(b)  $-2x + z = 2$

(c)  $-2x - z = -6$

(d)  $-x + 2y + z = 0$

(e)  $-x + 2y - z = -4$

(f)  $-x + 2y - z = 2$

$$\underbrace{2z - x^2 = 0}_{F(x,y,z)} \quad \nabla F = \langle -2x, 0, 2 \rangle$$

$$\nabla F(2,0,2) = (-4, 0, 2)$$

Tangent Plane:  $-4(x-2) + 0(y-0) + 2(z-2) = 0$

$$\therefore -4x + 2z + 4 = 0$$

$$-2x + z = -2$$

13. What is the equation for the line tangent to the curve of intersection of surfaces  $xyz = 1$  and  $x^2 + 2y^2 + 3z^2 = 6$  at  $(1, 1, 1)$ ?

(a)  $x = t + 1, y = 2t + 1, z = 2t + 1$

(b)  $x = t + 1, y = 2t + 1, z = 2t + 1$

(c)  $x = t + 1, y = -4t + 1, z = 2t + 1$

(d)  $x = 2t + 1, y = 2t + 1, z = 2t + 1$

(e)  $x = 2t + 1, y = -4t + 1, z = 2t + 1$

(f)  $x = 2t + 1, y = 4t + 1, z = 2t + 1$

$$\underbrace{xy_2 - 1 = 0}_{f(x,y,z)} \quad \nabla f = \langle yz, xz, yz \rangle$$

$$\nabla f(1,1,1) = \langle 1, 1, 1 \rangle$$

$$\underbrace{x^2 + 2y^2 + 3z^2 - 6 = 0}_{g(x,y,z)} \quad \nabla g = \langle 2xz, 4yz, 6z \rangle$$

$$\nabla g(1,1,1) = (2, 4, 6)$$

$$\langle 1, 1, 1 \rangle \times \langle 2, 4, 6 \rangle = \langle 2, -4, 2 \rangle$$

A line passing through  $(1, 1, 1)$  parallel to  $\langle 2, -4, 2 \rangle$ :

$$x = 1 + 2t \quad y = 1 - 4t \quad z = 1 + 2t$$

14. What are the absolute maximum and minimum values of  $f(x, y) = x^2 + 3x + y^2 - 4y + 1$  on the triangular region with vertices  $(0, 0), (0, 4), (2, 4)$ ?

(a) max = 11, min = 1

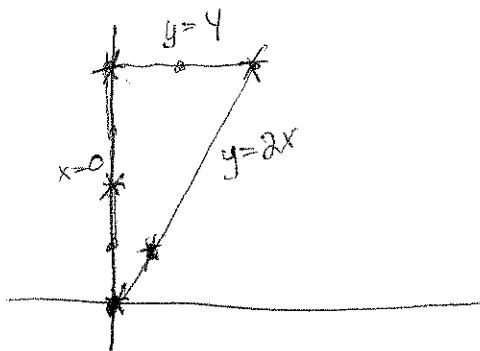
(b) max = 11, min = -3

(c) max = 13, min = 0

(d) max = 13, min = -3

(e) max = 11, min =  $\frac{-21}{4}$

(f) max = 13, min =  $\frac{5}{4}$



critical points:

$$\begin{aligned} f_x &= 2x + 3 = 0 & x = -\frac{3}{2} \\ f_y &= 2y - 4 = 0 & y = 2 \end{aligned} \quad \left. \begin{array}{l} x = -\frac{3}{2} \\ y = 2 \end{array} \right\} \text{outside the region}$$

vertices:  $(0,0), (0,4), (2,4)$

edges:  $x=0$ :  $f(0,y) = y^2 - 4y + 1$  derivative =  $2y - 4$   $y = 2$   $(0,2)$

$y=4$ :  $f(x,4) = x^2 + 3x + 1$  derivative =  $2x + 3$   $x = -\frac{3}{2}$  outside the region

$y=2x$ :  $f(x,2x) = x^2 + 3x + 4x^2 - 8x + 1 = 5x^2 - 5x + 1$   
derivative:  $10x - 5$   $x = \frac{1}{2}, y = 1$   $(\frac{1}{2}, 1)$

$f(0,0) = 1$   $f(0,4) = 1$   $f(2,4) = 11$   $f(0,2) = 4 - 8 + 1 = -3$   $f(\frac{1}{2}, 1) = \frac{1}{4} + \frac{3}{2} + 1 - 4 + 1 = 1 + 6 - 8 = -1$   
max min

15. What is the farthest distance of the points on the curve  $x^2 + xy + y^2 = 1$  from the origin?

- $$(a) \frac{1}{\sqrt{3}}$$

- (b)  $\frac{1}{3}$

- (c) 1

- (d)  $\sqrt{2}$

- (e)  $\sqrt{\frac{2}{3}}$

- (f) 4

$d^2 = x^2 + y^2$  we want to maximize  $d^2$  subject to

$$\underbrace{x^2 + xy + y^2}_{g(x,y)} = 1$$

$$\begin{cases} 2x = \lambda(2x+y) \\ 2y = \lambda(2y+x) \\ x^2 + xy + y^2 = 1 \end{cases}$$

$$\frac{2x}{2x+4} = \frac{2y}{2y+x}$$

$$so \quad 4xy + 2x^2 = 4xy + 2y^2$$

$$\text{so } x^2 = cy^2$$

$$so \quad x = \pm y$$

If  $x=y$ , from  $x^2 + xy + y^2 = 1$ , we get  $x^2 = \frac{1}{3}$ , so we get  $(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$   
 If  $x=-y$ , " " " we get  $x^2=1$ , so  $\underbrace{(1, -1), (-1, 1)}_{d^2=2}$   $\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right)$

16. If  $f_x = 2x - 4y$  and  $f_y = 2y - 4x$ , which of the following statements is true?

(a)  $f$  has only 1 saddle point.

(b)  $f$  has only 1 local maximum point.

(c)  $f$  has only 1 local minimum point.

(d)  $f$  has 1 saddle point and 1 local minimum.

(e)  $f$  has 1 saddle point and 1 local maximum.

(f)  $f$  has 1 local maximum and 1 local minimum.

$$\begin{array}{l} f_{xx} = 2 \quad f_{yy} = 2 \quad f_{xy} = -4 \quad \det \begin{pmatrix} 2 & -4 \\ -4 & 2 \end{pmatrix} < 0 \\ \left\{ \begin{array}{l} f_x = 0 \Rightarrow 2x - 4y = 0 \\ f_y = 0 \Rightarrow 2y - 4x = 0 \end{array} \right. \quad \left. \begin{array}{l} x=0 \quad y=0 \\ \text{saddle point} \end{array} \right\} \end{array}$$