solutions to the limit problems Homework # 5.

3. Find the limit, if it exists, or type N if it does not exist

$$\lim_{(x,y,z)\to(0,0,0)} \frac{3xy+5yz+5xz}{9x^2+25y^2+25z^2}$$

Answer: if we let $x=y$, $z=0$, so if we approach the origin along the line $x=y$, $z=0$, we get

$$\lim_{(x,y,z)\to(0,0,0)} \frac{3xy+5yz+5xz}{9x^2+25y^2+25z^2} = \lim_{x\to 0} \frac{3x^2+0+0}{9x^2+25x^2}$$

$$= \lim_{x\to 0} \frac{3x^2}{34x^2} = \frac{3}{34}.$$

Now if we let $x=0$, $y=z$, in other words if we approach the origin along the line $x=0$, $y=z$, then

$$\lim_{(x,y,z)\to(0,0,0)} \frac{3xy+5yz+5xz}{9x^2+25y^2+25z^2} = \lim_{y\to 0} \frac{5y^2}{25y^2+25z^2}$$

$$= \lim_{y\to 0} \frac{5}{50} = \frac{5}{50}$$

$\frac{3}{34} \neq \frac{5}{50}$, so the limit does not exist

(you can consider other paths too, for example $x=y=z$, the limit along this path is $\frac{13}{59}$)

or the path $x = y = 0$, the limit along this path is

$$\lim_{z \to 0} \frac{0}{25z^2} = 0$$)

4

(A) The limit $\lim_{(x,y)\to(0,0)} \frac{(x+19y)^2}{x^2+361y^2}$ does not exist: If we approach the origin along the x-axis, we get

$$\lim_{x\to 0} \frac{(x+0)^2}{x^2+0} = \lim_{x\to 0} \frac{x^2}{x^2} = 1$$

If we approach the origin along the y-axis, we get: $x=0$,

$$\lim_{y\to 0} \frac{(19y)^2}{361y^2} = \lim_{y\to 0} \frac{y^2}{y^2} = 1.$$

we cannot conclude the limit does not exist from this, but if we approach the origin along the line $x = y$, we get

$$\lim_{x\to 0} \frac{(x+19x)^2}{x^2+361x^2} = \lim_{x\to 0} \frac{400x^2}{362x^2} = \frac{400}{362} \neq 1.$$

---

4. B. $\lim_{(x,y)\to(0,0)} \frac{7x^3+y^3}{x^2+y^2} = ?$

The limit exists and is equal to zero.

we show that $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$; and $\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0$.

If we show these, we can conclude that:

$$\lim_{(x,y)\to(0,0)} \frac{7x^3+y^3}{x^2+y^2} = 7 \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} + \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0.$$

$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$:

We need to show that we can make $\left|\frac{x^3}{x^2+y^2}\right|$ as close to zero as we like if (x,y) is sufficiently close to $(0,0)$.

We look at the points (x,y) inside a disk of radius r around the origin, and see what we can say about $\left|\frac{x^3}{x^2+y^2}\right|$ for these points.

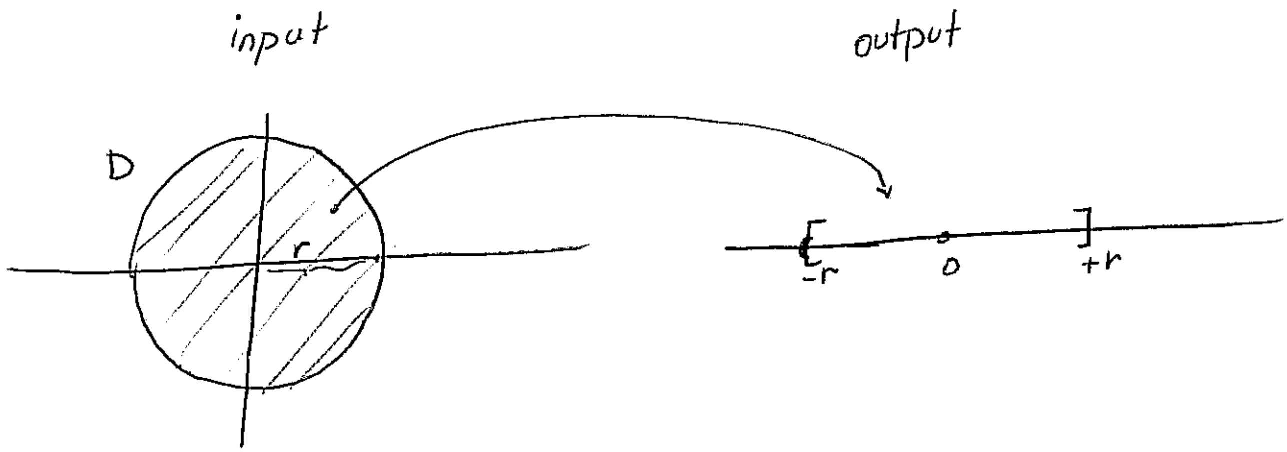
If D is a disk of radius r around $(0,0)$, and if (x,y) is in D, then $x^2+y^2 \le r$ and so $|x| \le r$,

so $$\left|\frac{x^3}{x^2+y^2}\right| = \left|\frac{x^2}{x^2+y^2}\right| |x|$$

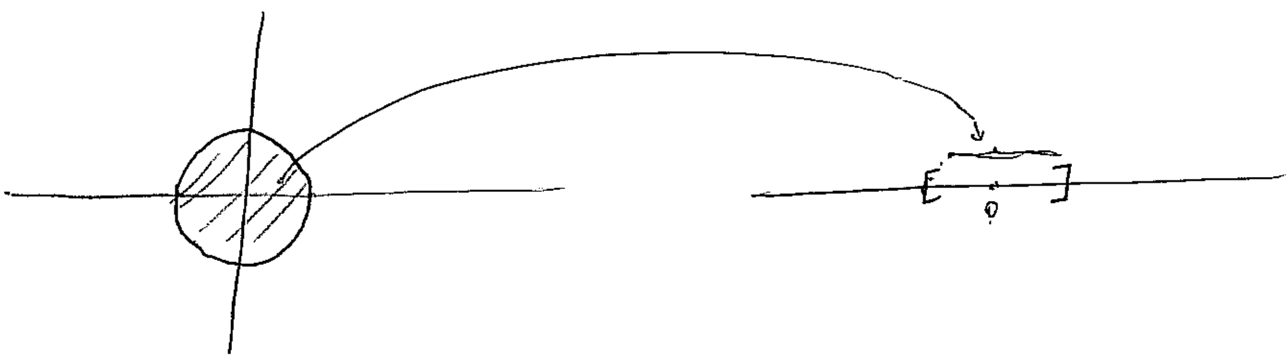
but for any two numbers x and y x^2 is always $\le x^2+y^2$

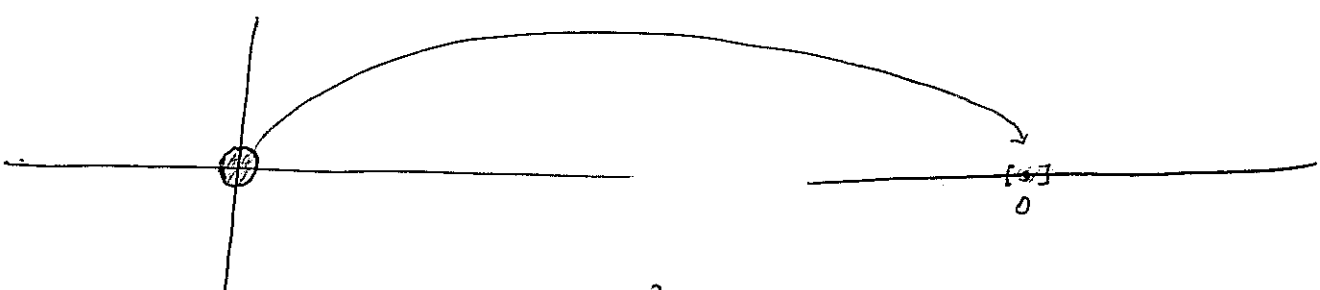
so $\frac{x^2}{x^2+y^2} \le 1$, so

$$\left|\frac{x^3}{x^2+y^2}\right| \leq 1 \cdot |x| = |x| \leq r.$$



If we now make the radius of the disk smaller and smaller, we can make sure that the output is getting as close to 0 as we like:





So by definition $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$, and a similar argument shows $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0.$

5 A. $\displaystyle\lim_{(x,y)\to(-2,2)} e^{\sqrt{x^2+y^2}} = e^{\sqrt{(-2)^2+2^2}} = e^{\sqrt{8}}$

B. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4x^2}{2x^2+2y^2}$ does not exist.

The reason is that if we approach the origin along the x-axis, $y=0$, so we get $\displaystyle\lim_{x\to 0} \frac{4x^2}{2x^2} = \lim_{x\to 0} \frac{4}{2} = 2.$

But if we approach the origin along the y-axis, $x=0$,

so $\displaystyle\lim_{y\to 0} \frac{0}{2y^2} = 0 \neq 2.$

---

6. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2+7y^2}{\sqrt{2x^2+7y^2+1}-1} = ?$

rationalizing, we get

$$\frac{2x^2+7y^2}{\sqrt{2x^2+7y^2+1}-1} = \frac{2x^2+7y^2}{\sqrt{2x^2+7y^2+1}-1} \cdot \frac{\sqrt{2x^2+7y^2+1}+1}{\sqrt{2x^2+7y^2+1}+1}$$

$$= \frac{(2x^2+7y^2)(\sqrt{2x^2+7y^2+1}+1)}{(2x^2+7y^2+1)-1}$$

$$= \sqrt{2x^2+7y^2+1}+1$$

therefore,

$$\lim_{(x,y)\to(0,0)} \frac{2x^2+7y^2}{\sqrt{2x^2+7y^2+1}-1} = \lim_{(x,y)\to(0,0)} \sqrt{2x^2+7y^2+1}+1 = \sqrt{1}+1$$

$$= 2.$$

---

11 Find the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(-5x+y)^2}{25x^2+y^2}$

1) along the x-axis : $y=0$ $\displaystyle\lim_{x\to 0} \frac{(-5x+0)^2}{25x^2+0} = \lim_{x\to 0} \frac{25x^2}{25x^2} = 1$

2) along the y-axis : $x=0$ $\displaystyle\lim_{y\to 0} \frac{(0+y)^2}{0+y^2} = \lim_{y\to 0} \frac{y^2}{y^2} = 1$

3) along the line $y=x$, $\displaystyle\lim_{x\to 0} \frac{(-5x+x)^2}{25x^2+x^2} = \lim_{x\to 0} \frac{16x^2}{26x^2} = \frac{16}{26}$

4) $\frac{16}{26} \neq 1$, so the limit does not exist.