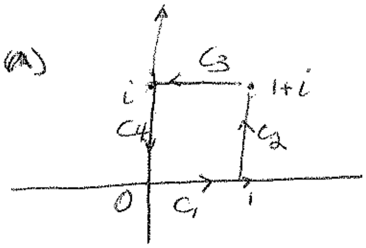
1. (a)



On C_1: $z=t$ $0\le t\le 1$, $\int_{C_1} \bar z\, dz = \int_0^1 t\, dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$

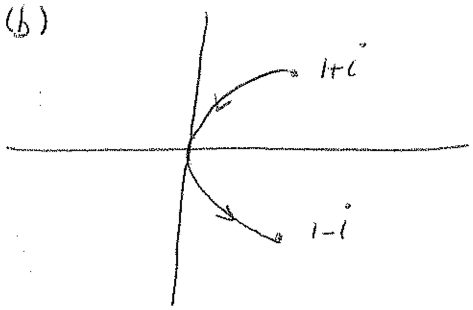
On C_2: $z=1+it$ $0\le t\le 1$, $\int_{C_2} \bar z dz = \int_0^1 (1-it)\, i\, dt = i+\frac{1}{2}$

On C_3: $z=1-t+i$ $0\le t\le 1$ $\int_{C_3} \bar z\, dz = \int_0^1 -(1-t-i)\, dt = i-\frac{1}{2}$

On C_4: $z = i(1-t)$ $0\le t\le 1$ $\int_{C_4} \bar z\, dz = \int_0^1 -i(1-t)\, -i\, dt = -\int_0^1 (1-t)\, dt = \frac{-1}{2}$

$\Rightarrow \int_C f(z)\, dz = \frac{1}{2} + (i+\frac{1}{2}) + (i-\frac{1}{2}) - \frac{1}{2} = 2i.$

(b)



$z = 1+e^{i\theta} \Rightarrow dz = i e^{i\theta} \Rightarrow \sqrt{z-1} = e^{i\theta/2}$ The principal part of θ goes from $(\frac{\pi}{2}, \pi)$ and $(-\pi, -\frac{\pi}{2})$

Therefore

$$\int_C \sqrt{z-1}\, dz = i\left(\int_{\frac{\pi}{2}}^{\pi} e^{\frac{3i\theta}{2}} d\theta + \int_{-\pi}^{-\frac{\pi}{2}} e^{\frac{3i\theta}{2}} d\theta \right) = \frac{2}{3} e^{\frac{3i\theta}{2}}\Big|_{\frac{\pi}{2}}^{\pi}$$

$$+ \frac{2}{3} e^{\frac{3i\theta}{2}}\Big|_{-\pi}^{-\frac{\pi}{2}} = \frac{2}{3}\left(e^{\frac{3i\pi}{2}} - e^{\frac{3i\pi}{4}} + e^{-\frac{3i\pi}{4}} - e^{-\frac{3\pi i}{4}}\right)$$

$$= \frac{-2}{3}(2+\sqrt{2})i$$

2. $$\int_{|z-i|=2} \frac{dz}{z^4-2z^3} = \int_{|z-i|=2} \frac{\frac{1}{z-2}}{z^3}\,dz = \frac{2\pi i}{2!}\left(\frac{1}{z-2}\right)''\Big|_{z=0} = \pi i \frac{2}{(z-2)^3}\Big|_{z=0}$$
$$= \frac{-\pi i}{4}$$

3. $f(z) = e^{z} + ie^{-z} = e^{x+iy} + ie^{-x-iy} = e^{x}(\cos y + i\sin y) + ie^{-x}(\cos y - i\sin y)$

$\Rightarrow |f(z)|^2 = (e^{x}\cos y + e^{-x}\sin y)^2 + (e^{x}\sin y + e^{-x}\cos y)^2$

$= e^{2x}(\cos^2 y + \sin^2 y) + e^{-2x}(\sin^2 y + \cos^2 y) + 4\cos y \sin y$

$= e^{2x} + e^{-2x} + 2\sin 2y$

Maximum : $\sin(2y) = 1$ and $\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, so $\underline{y = \frac{\pi}{4}}$

Also, if $g(x) = e^{2x} + e^{-2x}$, we need to find the maximum of $g(x)$. on $-1 \leq x \leq 1$

$g'(x) = 2(e^{2x} - e^{-2x}) = 0 \Rightarrow e^{2x}(1 - e^{-4x}) = 0 \Rightarrow x = 0$

So $x=0$ is a critical point of g, and we need to check the the points $x = \pm 1$ too. $g(1) = e^{2} + e^{-2} > g(0) = 2$
$g(-1) = e^{2} + e^{-2} > g(0) = 2$

So $\underline{x = \pm 1}$ are the maximum, and $\underline{x=0}$ is the minimum.

So $z = \pm 1 + \frac{i\pi}{4}$ is the maximum value of $|f(z)|$ which is on the boundary.

For the minimum, we have $x = 0$ and $\sin 2y = -1 \Rightarrow y = \frac{-\pi}{4}$

So $z = -\frac{i\pi}{4}$ is the point were min $|f(z)|$ happens and it is in the interior.

4. Let $f = u + iv$. Set $g(z) = e^{(1+i)f(z)} = e^{(1+i)(u+iv)} = e^{(u-v)+i(u+v)}$

$$= e^{u-v}\left[\cos(u+v) + i\sin(u+v)\right]$$

$\Rightarrow |g(z)| = e^{u-v}$. since $u \le v$, $u - v \le 0 \Rightarrow |g(z)| \le 1$. But $g(z)$ is an analytic function, so g should be a constant function

$\Rightarrow g'(z) = 0$

since $g(z) = e^{(1+i)f(z)}$, $g'(z) = (1+i)f'(z)\, e^{(1+i)F(z)} = 0$

$\Rightarrow f'(z) = 0 \Rightarrow f$ is a constant function

5. $f(z) = \dfrac{1}{z+1} - \dfrac{1}{z+4}$ so there are 3 regions:

D_1: $|z| < 1$

D_2: $1 < |z| < 4$

D_3: $4 < |z|$

on D_1: $\dfrac{1}{z+1} = \sum_{n=0}^{\infty} (-1)^n z^n$

$$\frac{1}{z+4} = \frac{1}{4} \cdot \frac{1}{\frac{z}{4}+1} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{4^{n+1}}$$

$$\Rightarrow \boxed{\frac{1}{z+1} - \frac{1}{z+4} = \sum_{n=0}^{\infty} (-1)^n z^n \left(1 - \frac{1}{4^{n+1}}\right)} \text{ on } D_1$$

on D_2: $\dfrac{1}{z+1} = \dfrac{1}{z} \dfrac{1}{\frac{1}{z}+1} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z}\right)^{n+1}$

$$\frac{1}{z+4} = \frac{1}{4} \frac{1}{\frac{z}{4}+1} = \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{4^{n+1}}$$

D_2

$$\Rightarrow \boxed{f(z) = \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{z^{n+1}} - \frac{z^n}{4^{n+1}}\right]}$$

in D_3: $\dfrac{1}{z+1} = \dfrac{1}{z} \dfrac{1}{\frac{1}{z}+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{n+1}}$

$$\frac{1}{z+4} = \frac{1}{z} \frac{1}{1+\frac{4}{z}} = \sum_{n=0}^{\infty} (-1)^n \frac{4^n}{z^{n+1}}$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{z^{n+1}} - \frac{4^n}{z^{n+1}}\right)$$

$$\boxed{= \sum_{n=0}^{\infty} (-1)^n \frac{1-4^n}{z^{n+1}}}$$

D_3