1. If $P = (2, 3, -1)$ and $Q = (5, 2, -3)$. Which vector is a unit vector in the direction of $\overrightarrow{PQ}$?

(a) $< \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}} >$

(b) $< \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}} >$

(c) $< \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}} >$

(d) $< \frac{3}{\sqrt{26}}, \frac{-1}{\sqrt{26}}, \frac{-4}{\sqrt{26}} >$

(e) $< \frac{3}{\sqrt{26}}, \frac{-1}{\sqrt{26}}, \frac{4}{\sqrt{26}} >$

(f) $< \frac{3}{\sqrt{26}}, \frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}} >$

$$\overrightarrow{PQ} = \langle 5-2, 2-3, -3-(-1) \rangle = \langle 3, -1, -2 \rangle$$

$$|\overrightarrow{PQ}| = \sqrt{3^2 + (-1)^2 + (-2)^2} = \sqrt{9+1+4} = \sqrt{14}$$

$$\frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|} = \frac{\langle 3, -1, -2 \rangle}{\sqrt{14}} = \left\langle \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}} \right\rangle$$

2. If $\mathbf{v} =< 2, 3, -1 >$, and $\mathbf{u} =< 1, -3, 7 >$, what is $\mathbf{v} \times \mathbf{u}$?

(a) $< 18, 15, -9 >$

(b) $< 18, -15, -9 >$

(c) $< 18, 13, -9 >$

(d) $< 18, -13, -9 >$

(e) $< -18, -15, 9 >$

(f) $< -18, -13, 9 >$

$$\vec{v} \times \vec{u} = \begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -3 & 7 \end{vmatrix} = \langle 18, -15, -9 \rangle$$

3. If $\mathbf{v} = < 1, 9, -3 >$. Which of the following vectors is perpendicular to $\mathbf{v}$?

(a) $< -1, -9, 3 >$

(b) $< -1, 9, 3 >$

(c) $< -3, 1, 2 >$

(d) $< 3, -1, 2 >$

(e) $< -3, \frac{1}{3}, 1 >$

$$\langle -3, 1, 2 \rangle \circ \langle 1, 9, -3 \rangle = (-3)(1) + (1)(9) + (2)(-3)$$
$$= -3 + 9 - 6 = 0.$$

4. What is the center and the radius of the sphere with equation

$$x^2 + y^2 + z^2 - 2x + 2y + 3z = \frac{-1}{4}?$$

(a) center $= (1, -1, 3)$; radius $= 4$

(b) center $= (1, 1, 3)$; radius $= 2$

(c) center $= (1, 1, 3)$; radius $= \frac{1}{2}$

(d) center $= (1, -1, \frac{3}{2})$; radius $= \frac{1}{2}$

(e) center $= (1, -1, \frac{-3}{2})$; radius $= 4$

(f) center $= (1, -1, -\frac{3}{2})$; radius $= 2$

$$x^2 + y^2 + z^2 - 2x + 2y + 3z = -\frac{1}{4}$$

so $$(x-1)^2 + (y+1)^2 + (z + \tfrac{3}{2})^2 = -\frac{1}{4} + 1 + 1 + \frac{9}{4}$$

$$= \frac{8}{4} + 2 = 4$$

$$= 2^2$$

center $= (1, -1, -\frac{3}{2})$

radius $= 2$

5. What is the distance from the point $Q(1,-1,2)$ to the plane with equation $2x+y-z=5$?

(a) 1

(b) $\frac{1}{\sqrt{6}}$

(c) 6

(d) $\sqrt{6}$

(e) $\frac{1}{6}$

$\vec{n} = \langle 2, 1, -1 \rangle$ normal vector for the plane

We next find a point on the plane :

$x = y = 0 \quad z = -5 \qquad P(0,0,-5)$

$\overrightarrow{PQ} = \langle 1, -1, 7 \rangle$

$\overrightarrow{PQ} \cdot \vec{n} = \langle 1, -1, 7 \rangle \cdot \langle 2, 1, -1 \rangle = 2 - 1 - 7 = -6$

$$\text{distance} = \frac{|\overrightarrow{PQ} \cdot \vec{n}|}{|\vec{n}|} = \frac{|-6|}{\sqrt{2^2+1^2+1^2}} = \frac{6}{\sqrt{6}} = \sqrt{6}$$

6. What is the area of the triangle with vertices $P(0, 1, 2)$, $Q(-1, 2, 2)$, and $R(4, -1, 0)$?

(a) $\sqrt{44}$

(b) $\sqrt{3}$

(c) $\sqrt{11}$

(d) $\sqrt{12}$

(e) $\sqrt{48}$

$$\vec{PQ} = \langle -1, 1, 0 \rangle$$

$$\vec{PR} = \langle 4, -2, -2 \rangle$$

$$\vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ 4 & -2 & -2 \end{vmatrix} = \langle -2, -2, -2 \rangle$$

$$|\vec{PQ} \times \vec{PR}| = \sqrt{2^2 + 2^2 + 2^2} = \sqrt{12}$$

$$\text{area} = \frac{|\vec{PQ} \times \vec{PR}|}{2} = \frac{\sqrt{12}}{2} = \sqrt{3}$$

7. What is the cosine of the angle between the two planes given by equations $2x + y + z = 4$, and $3x - y - 3z - 1 = 0$?

(a) $\frac{2}{\sqrt{102}}$

(b) $\frac{-2}{\sqrt{102}}$

(c) $\frac{4}{\sqrt{102}}$

(d) $\frac{-4}{\sqrt{102}}$

(e) $\frac{2}{\sqrt{114}}$

(f) $\frac{-2}{\sqrt{114}}$

$\vec{n}_1 = \langle 2, 1, 1 \rangle \qquad |\vec{n}_1| = \sqrt{6}$

$\vec{n}_2 = \langle 3, -1, -3 \rangle \qquad |n_2| = \sqrt{19}$

$\vec{n}_1 \cdot \vec{n}_2 = 2 \cdot 3 + 1\,(-1) + 1\,(-3) = 2$

$\cos\theta = \dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1|\,|\vec{n}_2|} = \dfrac{2}{\sqrt{6}\,\sqrt{19}} = \dfrac{2}{\sqrt{114}}$

8. What is the equation of a plane passing through the point $P(5, 7, -1)$ and parallel to the plane $x + y + 2z = 1$?

(a) $x + y + 2z = 10$

(b) $x - y + 2z = -4$

(c) $2x + 2z = 6$

(d) $-x + y + z = 1$

(e) $x - y - 3z = 1$

$\vec{n} = \langle 1, 1, 2 \rangle$

equation:

$\langle x-5, y-7, z+1 \rangle \cdot \langle 1, 1, 2 \rangle = 0$

So $(x-5) + (y-7) + 2(z+1) = 0$

So $x + y + 2z = 10$

9. Which of the following equations is the vector equation of the line of intersection of the two planes given by $x + y = 2$ and $x - y - 3z = 0$?

(a) $t < -5, 1, -2 > + < 0, 2, \frac{3}{2} >$

(b) $t < 5, -1, 2 > + < 3, 0, 1 >$

(c) $t < -5, 1, -2 > + < 1, 1, 0 >$

(d) $t < -3, -3, -2 > + < 0, 2, \frac{3}{2} >$

(e) $t < -3, 3, -2 > + < 0, 2, 0 >$

(f) $t < 3, -3, 2 > + < 1, 1, 0 >$

$\vec{n}_1 = \langle 1, 1, 0 \rangle$

$\vec{n}_2 = \langle 1, -1, -3 \rangle$

$$\vec{V} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 1 & -1 & -3 \end{vmatrix} = \langle -3, 3, -2 \rangle$$

10. Assume $\mathbf{u}(t)$ and $\mathrm{w}(t)$ are two vector functions such that

$$\mathbf{u}(t) = < t^2, \sin t, 1+2t >,$$
$$\mathrm{w}(0) = < 1, 0, -2 >,$$
$$\mathrm{w}'(0) = < 1, 1, 3 > .$$

If $f(t) = \mathbf{u}(t) \cdot \mathbf{w}(t)$, what is $f'(0)$?

(a) -1

(b) 1

(c) 0

(d) 2

(e) -2

$$f(t) = \vec{u}(t) \circ \vec{w}(t)$$

$$f'(t) = (\vec{u}(t) \cdot \vec{w}(t))' = \vec{u}'(t) \cdot \vec{w}(t) + \vec{u}(t) \cdot \vec{w}'(t)$$

so $f'(0) = \vec{u}'(0) \cdot \vec{w}(0) + \vec{u}(0) \cdot \vec{w}'(0)$

$$\vec{u}(0) = \langle 0, 0, 1 \rangle$$
$$\vec{u}'(0) = \langle 0, 1, 2 \rangle$$

so $f'(0) = \langle 0,1,2 \rangle \cdot \langle 1,0,-2 \rangle + \langle 0,0,1 \rangle \cdot \langle 1,1,3 \rangle$

$$= -4 + 3 = -1$$

11. What is the parametric equation of the tangent line to the curve traced by $\mathbf{r}(t) = (\cos t)\,\mathbf{i} + e^t\,\mathbf{j} + 2t\,\mathbf{k}$ at $t = 0$?

(a) $x = s;\ \ y = 1 - s;\ \ z = 2$

(b) $x = s;\ \ y = 1 + s;\ \ z = 2s$

(c) $x = 1;\ \ y = 1 - s;\ \ z = 2s$

(d) $x = 1;\ \ y = 1 + s;\ \ z = 2s$

(e) $x = s;\ \ y = 1 + s;\ \ z = 2$

$$\vec{r}\,'(t) = -\sin(t)\,\vec{i} + e^{t}\,\vec{j} + 2\,\vec{k}$$

$$\vec{r}\,'(0) = \vec{j} + 2\vec{k} = \langle 0, 1, 2 \rangle$$

$$r(0) = \vec{i} + \vec{j} = \langle 1, 1, 0 \rangle$$

The parametric equation of the line =

$$x = 1 + 0 \cdot s \qquad y = 1 + 1 \cdot s \qquad z = 0 + 2 \cdot s$$

or

$$x = 1 \qquad y = 1 + s \qquad z = 2s$$

12. The Assume $\mathbf{r}(t)$ is the position vector of a particle in space. If the acceleration vector is $\mathbf{a}(t) = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j}$, the initial position $\mathbf{r}(0) = -\mathbf{i} + 5\mathbf{k}$, and the initial velocity vector is $\mathbf{v}(0) = \mathbf{j}$. What is the position vector at $t = \frac{\pi}{2}$?

(a) $-2\mathbf{i} - (1+\pi)\mathbf{j} + \mathbf{k}$

(b) $(-2+\pi)\mathbf{i} + \mathbf{k}$

(c) $-(1+\pi)\mathbf{j} + 5\mathbf{k}$

(d) $(2+\pi)\mathbf{i} + \mathbf{k}$

(e) $-\mathbf{i} + \mathbf{j} + 5\mathbf{k}$

(f) $(\pi - 1)\mathbf{j} + 5\mathbf{k}$

$\vec{a}(t) = \vec{v}'(t)$ so $\vec{v}(t) = \int \vec{a}(t)\,dt$

$= \int (\cos t)\,\vec{i} + (\sin t)\,\vec{j}\,dt$

$= +\sin t\,\vec{i} - \cos t\,\vec{j} + \vec{C}_1$

$t = 0$ $\quad \vec{j} = \vec{v}(0) = \sin(0)\,\vec{i} - \cos(0)\,\vec{j} + \vec{C}_1 = -\vec{j} + \vec{C}_1$

so $\vec{C}_1 = 2\vec{j}$. so $\vec{v}(t) = (\sin t)\,\vec{i} + (-\cos t + 2)\,\vec{j}$

$\vec{v}(t) = \vec{r}\,'(t)$ so $\vec{r}(t) = \int \vec{v}(t)\,dt = \int (\sin t)\,\vec{i} + (-\cos t + 2)\,\vec{j}\,d$

$= (-\cos(t))\,\vec{i} + (-\sin t + 2t)\,\vec{j} + \vec{C}$

$\vec{r}(0) = -\vec{i} + 5\vec{k}$ so $-\vec{i} + 5\vec{k} = -\vec{i} + \vec{C}_2$ so $\vec{C}_2 = 5\vec{k}$

so $\vec{r}(t) = (-\cos t)\,\vec{i} + (-\sin t + 2t)\,\vec{j} + 5\vec{k}$

13. What is the point on the curve

$$\mathbf{r}(t) = (3\sin t)\,\mathbf{i} - (3\cos t)\,\mathbf{j} + 4t\,\mathbf{k}$$

at a distance $\frac{5\pi}{2}$ units along the curve from the point $(0, -3, 0)$ in the direction of increasing arc length?

(a) $(0, 3, 4\pi)$

(b) $(3, 0, 2\pi)$

(c) $(-3, 0, -2\pi)$

(d) $(0, 3, 2\pi)$

(e) $(0, -3, -4\pi)$

(f) $(3, 0, 4\pi)$

$\vec{r}(0) = \langle 0, -3, 0 \rangle$

$s(t) = \int_0^t |\vec{v}(\tau)|\, d\tau$

$\vec{v}(t) = \vec{r}'(t) = \langle 3\cos t, +3\sin t, 4 \rangle$

$|\vec{v}(t)| = \sqrt{9\cos^2(t) + 9\sin^2(t) + 16} = \sqrt{9+16} = 5$

so

$s(t) = \int_0^t 5\, d\tau = 5\tau \Big|_0^t = 5t$

$5t = \frac{5\pi}{2}$ so $t = \frac{\pi}{2}$

so $\vec{r}(\frac{\pi}{2}) = \langle 3, 0, 2\pi \rangle$

Hand-Graded Question [12 points]

14. Assume that

$$\mathbf{r}(t) = (2t-1)\,\mathbf{i} + (t^2-1)\,\mathbf{j} + (4t-3)\,\mathbf{k}$$

is the position vector of a particle moving in space.

(a) [3 points] Find the velocity vector at $t = 1$.

$\vec{v}(t) = \vec{r}\,'(t) = 2\vec{i} + 2t\vec{j} + 4\vec{k}$

velocity vector at $t=1$: $\vec{v}(1) = 2\vec{i} + 2\vec{j} + 4\vec{k}$

(b) [3 points] Find the acceleration vector at $t = 1$.

$\vec{a}(t) = \vec{v}\,'(t) = 2\vec{j}$

acceleration vector at $t=1$ is $2\vec{j}$

(c) [2 points] What is the speed at $t = 1$?

speed = magnitude of velocity

$= |2\vec{i} + 2t\vec{j} + 4\vec{k}| = \sqrt{4+4t^2+16} = \sqrt{20+4t^2}$

speed at $t=1$: $\sqrt{24}$

(d) [4 points] What is the angle between the position vector and the velocity vector at $t = 1$?

If θ is the angle between $\vec{r}(1)$ and $\vec{v}(1)$, then

$$\cos(\theta) = \frac{\vec{r}(1) \cdot \vec{v}(1)}{|\vec{r}(1)|\ |\vec{v}(1)|}$$

$\vec{r}(1) = \langle 1, 0, 1 \rangle$ $\qquad |\vec{r}(1)| = \sqrt{2}$

$\vec{v}(1) = \langle 2, 2, 4 \rangle$ $\qquad |\vec{v}(1)| = \sqrt{24}$

$\vec{v}(1) \cdot \vec{r}(1) = 1 \cdot 2 + 0 \cdot 2 + 1 \cdot 4 = 6$

So

$$\cos\theta = \frac{6}{\sqrt{2}\sqrt{24}} = \frac{6}{\sqrt{48}} = \frac{6}{\sqrt{6}\sqrt{8}} = \frac{\sqrt{3}}{2}$$

$\Rightarrow \theta = 30^{\circ}$

Hand-Graded Question [10 points]

15. Let $\mathbf{u} =< 0, 4, 1 >$ and $\mathbf{v} =< 3, -2, 1 >$.

(a) [5 points] Find the vector projection of $\mathbf{u}$ onto $\mathbf{v}$.

$$\operatorname{proj}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \vec{v}$$

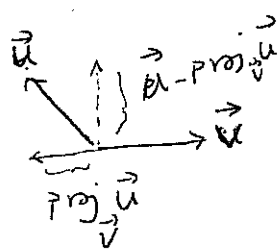
$$\vec{u} \cdot \vec{v} = 0 \cdot 3 + 4 \cdot (-2) + 1 \cdot 1 = -7$$

$$|\vec{v}|^2 = 3^2 + (-2)^2 + 1^2 = 14$$

so $\operatorname{proj}_{\vec{v}} \vec{u} = \frac{-7}{14} \vec{v} = -\frac{1}{2} \langle 3, -2, 1 \rangle = \langle -\frac{3}{2}, 1, -\frac{1}{2} \rangle$

(b) [5 points] Write $\mathbf{u}$ as the sum of two vectors: one perpendicular to $\mathbf{v}$ and one parallel to $\mathbf{v}$.

$\vec{u} - \operatorname{proj}_{\vec{v}} \vec{u}$ is perpendicular to $\vec{v}$.



and

$$\vec{u} = \underbrace{(\operatorname{proj}_{\vec{v}} \vec{u})}_{\text{parallel to } \vec{v}} + \underbrace{(\vec{u} - \operatorname{proj}_{\vec{v}} \vec{u})}_{\text{perpendicular to } \vec{v}}$$

so

$$\vec{u} = \langle -\frac{3}{2}, 1, -\frac{1}{2} \rangle + \langle \frac{3}{2}, 3, \frac{3}{2} \rangle$$