

NON-UNIRULEDNESS RESULTS FOR SPACES OF RATIONAL CURVES IN HYPERSURFACES

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ABSTRACT. We prove that the sweeping components of the space of smooth rational curves in a smooth hypersurface of degree d in \mathbf{P}^n are not uniruled if $(n+1)/2 \leq d \leq n-3$. We also show that for any $e \geq 1$, the space of smooth rational curves of degree e in a general hypersurface of degree d in \mathbf{P}^n is not uniruled roughly when $d \geq e\sqrt{n}$.

1. INTRODUCTION

Throughout this paper, we work over an algebraically closed field of characteristic zero \mathbf{k} . Let X be a smooth hypersurface of degree d in \mathbf{P}^n , and let $\text{Hilb}_{et+1}(X)$ be the Hilbert scheme parametrizing subschemes of X with Hilbert polynomial $et+1$. Denote by $R_e(X)$ the closure of the open subscheme of $\text{Hilb}_{et+1}(X)$ parametrizing smooth rational curves of degree e . Following [12], we call an irreducible component R of $R_e(X)$ a *sweeping component* if the curves parametrized by its points sweep out X or equivalently, if for a general point $[C]$ in R , $N_{C/X}$ is globally generated.

In this paper, we consider the birational geometry of the sweeping components of $R_e(X)$, specifically, we are interested in the following question: for which values of n, d , and e , does $R_e(X)$ have non-uniruled sweeping components? A projective variety Y of dimension m is called uniruled if there is a variety Z of dimension $m-1$ and a dominant rational map $Z \times \mathbf{P}^1 \dashrightarrow Y$. Our original motivation for this study comes from the question of whether or not general Fano hypersurfaces of low index are unirational.

We prove the following:

Theorem 1.1. *Let X be any smooth hypersurface of degree d in \mathbf{P}^n , $(n+1)/2 \leq d \leq n-3$. Then the sweeping components of $R_e(X)$ are all non-uniruled.*

Note that if $d \leq n-1$, or if $d = n$ and $e \geq 2$, $R_e(X)$ has at least one sweeping component. Also note that when $d \leq (n+4)/2$, $(d, n) \neq (3, 3)$ and X is general, $R_e(X)$ is irreducible (see [6] and [2]), and it is conjectured that the same holds for general Fano hypersurface of dimension at least three [2].

The proof of Theorem 1.1 shows that when $d = n-2$ a sweeping component R of $R_e(X)$ is non-uniruled if the normal bundle of a general curve parametrized by R is balanced (see Proposition 3.2). In the special case when $n = 5$ and $d = 3$, $R_e(X)$ is irreducible but it is not balanced. Modifying the proof of Theorem 1.1, we give a new proof of the following theorem:

Theorem 1.2 (de Jong-Starr [3]). *If X is a general cubic fourfold, then $R_e(X)$ is not uniruled when $e > 5$ is an odd integer, and the general fibers of the MRC fibration of*

any desingularization of $R_e(X)$ are at most 1-dimensional when $e > 4$ is an even integer.

The questions which remain are first, what happens when $d = n - 1$, or $d = n$ and $e \geq 2$? Second, how small can d be for $R_e(X)$ to be non-uniruled? When $d = n$, the uniruledness of the sweeping subvarieties of $R_e(X)$ has been studied in [1]. It is shown that if $d = n$ and $e \leq n$, a subvariety of $R_e(X)$ is non-uniruled if the curves parametrized by its points sweep out X or a divisor in X .

When $d^2 \leq n$, X is rationally simply connected (see [11] and [4]), and in particular $R_e(X)$ is uniruled. There are evidences which suggest that when $d^2 + d \geq 2n + 2$, $R_e(X)$ is non-uniruled for general X , but this is known to be true only for $e = 1$. The following is one such evidence:

Theorem 1.3 (J. Starr [12]). *Let $X \subset \mathbf{P}^n$ be a general hypersurface of degree d , and let $\overline{\mathcal{M}}_{0,0}(X, e)$ be the Kontsevich moduli space parametrizing rational curves of degree e on X . If $d < \min(n - 6, \frac{n+1}{2})$ and $d^2 + d \geq 2n + 2$, then for every $e > 0$ the canonical divisor of $\overline{\mathcal{M}}_{0,0}(X, e)$ is big.*

One cannot directly conclude from the theorem above that the coarse moduli space $\overline{M}_{0,0}(X, e)$ is also of general type when d satisfies the inequalities given in the statement of the theorem. In [12], it is conjectured that when $d + e \leq n$ and d is in the range given in Theorem 1.3, $\overline{M}_{0,0}(X, e)$ has at worst canonical singularities, and it is shown that assuming this conjecture, the above theorem implies $\overline{M}_{0,0}(X, e)$ and hence $R_e(X)$ are of general type when $d + e \leq n$.

In Section 4, we show:

Theorem 1.4. *Let $X \subset \mathbf{P}^n$ ($n \geq 12$) be a general hypersurface of degree d , and let $m \geq 1$ be an integer. If a general smooth rational curve C in X is m -normal (i.e. the global sections of $\mathcal{O}_{\mathbf{P}^n}(m)$ maps surjectively to those of $\mathcal{O}_{\mathbf{P}^n}(m)|_C$), and if*

$$d^2 + (2m + 1)d \geq (m + 1)(m + 2)n + 2,$$

then $R_e(X)$ is not uniruled.

Since every smooth rational curve of degree $e \geq 3$ in \mathbf{P}^n is $(e - 2)$ -normal, we get:

Corollary 1.5. *Let X be a general hypersurface of degree d in \mathbf{P}^n , $n \geq 12$. If $e \geq 3$ is an integer, and if*

$$d^2 + (2e - 3)d \geq e(e - 1)n + 2,$$

then $R_e(X)$ is not uniruled.

As it will be explained in the last section, we expect that better upper bounds exist on the regularity of general smooth rational curves contained in a general smooth hypersurface of degree d in \mathbf{P}^n , so the bound in Corollary 1.5 could be possibly improved.

Acknowledgments. The author would like to thank Izzet Coskun, Mohan Kumar, Mike Roth, and Jason Starr for many helpful conversations.

2. A CONSEQUENCE OF UNIRULEDNESS

In this section, we prove a proposition, analogous to the existence of free rational curves on non-singular uniruled varieties, for varieties whose spaces of smooth rational curves are uniruled.

For a morphism $f : Y \rightarrow X$ between smooth varieties, by the *normal sheaf of f* we will mean the cokernel of the induced map on the tangent bundles $T_Y \rightarrow f^*T_X$.

If Y is an irreducible projective variety, and if \tilde{Y} is a desingularization of Y , then the maximal rationally connected (MRC) fibration of \tilde{Y} is a smooth morphism $\pi : Y^0 \rightarrow Z$ from an open subset $Y^0 \subset \tilde{Y}$ such that the fibers of π are all rationally connected, and such that for a very general point $z \in Z$, any rational curve in \tilde{Y} intersecting $\pi^{-1}(z)$ is contained in $\pi^{-1}(z)$. The MRC fibration of any smooth variety exists and is unique up to birational equivalences [9].

Let Y be an irreducible projective variety, and assume the fiber of the MRC fibration of \tilde{Y} at a general point is m -dimensional. Then it follows from the definition that there is an irreducible component Z of $\text{Hom}(\mathbf{P}^1, Y)$ such that the map $\mu_1 : Z \times \mathbf{P}^1 \rightarrow Y$ defined by $\mu_1([g], b) = g(b)$ is dominant and the image of the map $\mu_2 : Z \times \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow Y \times Y$ defined by $\mu_2([g], b_1, b_2) = (g(b_1), g(b_2))$ has dimension $\geq \dim Y + m$.

Proposition 2.1. *Let $X \subset \mathbf{P}^n$ be a nonsingular projective variety. If an irreducible sweeping component R of $R_e(X)$ is uniruled, then there exist a smooth rational surface S with a dominant morphism $\pi : S \rightarrow \mathbf{P}^1$ and a generically finite morphism $f : S \rightarrow X$ with the following two properties:*

- (i) *If C is a general fiber of π , then $f|_C$ is a closed immersion onto a smooth curve parametrized by a general point of R .*
- (ii) *If N_f denotes the normal sheaf of f , then π_*N_f is globally generated.*

Moreover, if the fiber of the MRC fibration of a desingularization of R at a general point is at least m -dimensional, then there are such S and f with the additional property that π_*N_f has an ample subsheaf of rank $= m - 1$.

Proof. Let $U \subset R \times X$ be the universal family over R . Since R is uniruled, there exist a quasi-projective variety Z and a dominant morphism $\mu : Z \times \mathbf{P}^1 \rightarrow R$. Let $V \subset Z \times \mathbf{P}^1 \times X$ be the pullback of the universal family to $Z \times \mathbf{P}^1$, and denote by $q : V \rightarrow Z \times X$ and $p : V \rightarrow Z$ the projection maps.

Consider a desingularization $g : \tilde{V} \rightarrow V$, and let $\tilde{q} = q \circ g$ and $\tilde{p} = p \circ g$. Denote the fibers of p and \tilde{p} over z by S and \tilde{S} respectively. Let $f : S \rightarrow X$ be the restriction of q to S , and let $\tilde{f} = f \circ g : \tilde{S} \rightarrow X$. Since z is general, by generic smoothness, \tilde{S} is a smooth surface whose general fiber over \mathbf{P}^1 is a smooth connected rational curve. We claim that \tilde{S} and \tilde{f} satisfy the properties of the theorem. The first property is clearly satisfied.

To show the second property is satisfied, we consider the Kodaira-Spencer map associated to \tilde{V} at a general point $z \in Z$. Denote by $N_{\tilde{q}}$ the normal sheaf of the map \tilde{q} . We get a sequence of maps

$$T_{Z,z} \rightarrow H^0(\tilde{S}, \tilde{p}^*T_Z|_{\tilde{S}}) \rightarrow H^0(\tilde{S}, \tilde{q}^*T_{X \times Z}|_{\tilde{S}}) \rightarrow H^0(\tilde{S}, N_{\tilde{q}}|_{\tilde{S}}).$$

Let b be a general point of \mathbf{P}^1 . Composing the above map with the projection map $T_{Z \times \mathbf{P}^1, (z,b)} \rightarrow T_{Z,z}$, we get a map $T_{Z \times \mathbf{P}^1, (z,b)} \rightarrow H^0(\tilde{S}, N_{\tilde{q}}|_{\tilde{S}})$. Note that if $N_{\tilde{f}}$ denotes

the normal sheaf of \tilde{f} , then $N_{\tilde{q}|\tilde{S}}$ is naturally isomorphic to $N_{\tilde{f}}$. Also, if C is the fiber of $\pi : \tilde{S} \rightarrow \mathbf{P}^1$ over b , then we have a short exact sequence

$$0 \rightarrow N_{C/\tilde{S}} \rightarrow N_{\tilde{f}(C)/X} \rightarrow N_{\tilde{f}|C} \rightarrow 0.$$

So we get a commutative diagram

$$\begin{array}{ccccc} T_{Z \times \mathbf{P}^1, (z, b)} & \longrightarrow & T_{Z, z} & \longrightarrow & H^0(\tilde{S}, N_{\tilde{f}}) \\ \downarrow d\mu_{(z, b)} & & & & \downarrow \\ T_{R, [\tilde{f}(C)]} = H^0(\tilde{f}(C), N_{\tilde{f}(C)/X}) & \longrightarrow & & \longrightarrow & H^0(C, N_{\tilde{f}|C}) \end{array}$$

Since μ is dominant, and since R is sweeping and therefore generically smooth, $d\mu_{(z, b)}$ is surjective. Since the bottom row is also surjective, the map $H^0(\tilde{S}, N_{\tilde{f}}) \rightarrow H^0(C, N_{\tilde{f}|C})$ is surjective as well. Thus $\tilde{\pi}_* N_{\tilde{f}}$ is globally generated.

Suppose now that R is uniruled and that the general fibers of the MRC fibration of R are at least m -dimensional. Let $\dim R = r$. Then there exists a morphism $\mu_1 : Z \times \mathbf{P}^1 \rightarrow R$ such that the image of

$$\mu_2 : Z \times \mathbf{P}^1 \times \mathbf{P}^1 \rightarrow R \times R$$

$$\mu_2(z, b_1, b_2) = (\mu_1(z, b_1), \mu_1(z, b_2))$$

has dimension $\geq r + m$. If \tilde{S} and \tilde{f} are as before, and if C_1 and C_2 denote the fibers of π over general points b_1 and b_2 of \mathbf{P}^1 , then the image of the map

$$d\mu_2 : T_{Z \times \mathbf{P}^1 \times \mathbf{P}^1, (z, b_1, b_2)} \rightarrow T_{R \times R, ([\tilde{f}(C_1)], [\tilde{f}(C_2)])} = H^0(C_1, N_{\tilde{f}(C_1)/X}) \oplus H^0(C_2, N_{\tilde{f}(C_2)/X})$$

is at least $(r + m)$ -dimensional. The desired result now follows from the following commutative diagram

$$\begin{array}{ccccc} T_{Z \times \mathbf{P}^1 \times \mathbf{P}^1, (z, b_1, b_2)} & \longrightarrow & T_{Z, z} & \longrightarrow & H^0(\tilde{S}, N_{\tilde{f}}) \\ \downarrow (d\mu_2)_{(z, b_1, b_2)} & & & & \downarrow \\ T_{R \times R, ([\tilde{f}(C_1)], [\tilde{f}(C_2)])} & \longrightarrow & & \longrightarrow & H^0(C_1, N_{\tilde{f}|C_1}) \oplus H^0(C_2, N_{\tilde{f}|C_2}), \end{array}$$

and the observation that the kernel of the bottom row is 2-dimensional. \square

The above proposition will be enough for the proof of Theorem 1.1, but to prove Theorem 1.2 in the even case, we will need a slightly stronger variant. Let $f : Y \rightarrow X$ be a morphism between smooth varieties, and let N_f be the normal sheaf of f

$$0 \rightarrow T_Y \rightarrow f^*T_X \rightarrow N_f \rightarrow 0.$$

Suppose there is a dominant map $\pi : Y \rightarrow \mathbf{P}^1$, and let M be the image of the map induced by π on the tangent bundles $T_Y \rightarrow \pi^*T_{\mathbf{P}^1}$. Consider the push-out of the above sequence by the map $T_Y \rightarrow M$

$$\begin{array}{ccccccc}
0 & \longrightarrow & T_Y & \longrightarrow & f^*T_X & \longrightarrow & N_f \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow = \\
0 & \longrightarrow & M & \longrightarrow & N_{f,\pi} & \longrightarrow & N_f \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & &
\end{array}$$

The sheaf $N_{f,\pi}$ in the above diagram will be referred to as the *normal sheaf of f relative to π* .

Property (ii) of Proposition 2.1 says that $H^0(S, N_f) \rightarrow H^0(C, N_f|_C)$ is surjective. An argument parallel to the proof of Proposition 2.1 shows the following:

Proposition 2.2. *Let X be as in Proposition 2.1. Then property (ii) can be strengthened as follows:*

(ii') *If N_f denotes the normal sheaf of f , and if $N_{f,\pi}$ denotes the normal sheaf of f relative to π , then the composition of the maps*

$$H^0(S, N_{f,\pi}) \rightarrow H^0(C, N_{f,\pi}|_C) \rightarrow H^0(C, N_f|_C)$$

is surjective for a general fiber C of π .

Moreover, if the general fibers of the MRC fibration of a desingularization of R are at least m -dimensional, then there are S and f with properties (i) and (ii') such that the image of the map

$$H^0(S, N_{f,\pi} \otimes I_C) \rightarrow H^0(C, (N_f \otimes I_C)|_C)$$

is at least $(m-1)$ -dimensional.

3. PROOF OF THEOREM 1.1

Throughout this section, X will be a smooth hypersurface of degree $d \leq n-3$ in \mathbf{P}^n . Assume that a sweeping component R of $R_e(X)$ is uniruled. Then we apply Proposition 2.1 to show that $d < (n+1)/2$.

Proposition 3.1. *Let S and f be as in Proposition 2.1. If C is a general fiber of $\pi : S \rightarrow \mathbf{P}^1$ and I_C is the ideal sheaf of C in S , then the restriction map*

$$H^0(S, f^*\mathcal{O}_X(2d-n-1) \otimes I_C^\vee) \rightarrow H^1(C, f^*\mathcal{O}_X(2d-n-1) \otimes I_C^\vee|_C)$$

is zero.

If the above map is zero, then $H^0(S, f^*\mathcal{O}_X(2d-n-1)) = H^0(S, f^*\mathcal{O}_X(2d-n-1) \otimes I_C^\vee)$. Thus,

$$\begin{aligned}
H^0(\mathbf{P}^1, \pi_* f^*\mathcal{O}_X(2d-n-1)) &= H^0(\mathbf{P}^1, \pi_*(f^*\mathcal{O}_X(2d-n-1) \otimes I_C^\vee)) \\
&= H^0(\mathbf{P}^1, (\pi_* f^*\mathcal{O}_X(2d-n-1)) \otimes \mathcal{O}_{\mathbf{P}^1}(1)),
\end{aligned}$$

which is only possible if $H^0(\mathbf{P}^1, \pi_* f^*\mathcal{O}_X(2d-n-1)) = 0$. So $H^0(S, f^*\mathcal{O}_X(2d-n-1)) = 0$ and $d < (n+1)/2$.

Proof of Proposition 3.1. Let ω_S be the canonical sheaf of S . By Serre duality, it suffices to show that if S and f satisfy the properties of Proposition 2.1, then the restriction map

$$H^1(S, f^*\mathcal{O}_X(n+1-2d) \otimes \omega_S) \rightarrow H^1(C, f^*\mathcal{O}_X(n+1-2d) \otimes \omega_S|_C)$$

is surjective. Let N be the normal sheaf of the map $f : S \rightarrow X$, and let N' be the normal sheaf of the map $S \rightarrow \mathbf{P}^n$. Since the normal bundle of X in \mathbf{P}^n is isomorphic to $\mathcal{O}_X(d)$, we get a short exact sequence

$$(1) \quad 0 \rightarrow N \rightarrow N' \rightarrow f^*\mathcal{O}_X(d) \rightarrow 0.$$

Taking the $(n-3)$ -rd exterior power of this sequence, we get the following short exact sequence

$$0 \rightarrow \bigwedge^{n-3} N \otimes f^*\mathcal{O}_X(-d) \rightarrow \bigwedge^{n-3} N' \otimes f^*\mathcal{O}_X(-d) \rightarrow \bigwedge^{n-4} N \rightarrow 0.$$

For an exact sequence of sheaves of \mathcal{O}_S -modules $0 \rightarrow E \rightarrow F \rightarrow M \rightarrow 0$ with E and F locally free of ranks e and f , there is a natural map of sheaves

$$\bigwedge^{f-e-1} M \otimes \bigwedge^e E \otimes (\bigwedge^f F)^\vee \rightarrow M^\vee$$

which is defined locally at a point $s \in S$ as follows: assume $\gamma_1, \dots, \gamma_{f-e-1} \in M_s$, $\alpha_1, \dots, \alpha_e \in E_s$, and $\phi : \bigwedge^f F_s \rightarrow \mathcal{O}_{S,s}$; then for $\gamma \in M_s$, we set $\gamma_{f-e} = \gamma$, and we define the map to be

$$\gamma \mapsto \phi(\tilde{\gamma}_1 \wedge \tilde{\gamma}_2 \wedge \dots \wedge \tilde{\gamma}_{f-e} \wedge \alpha_1 \wedge \dots \wedge \alpha_e)$$

where $\tilde{\gamma}_i$ is any lifting of γ_i in F_s . Clearly, this map does not depend on the choice of the liftings, and thus it is defined globally. So from the short exact sequence $0 \rightarrow T_S \rightarrow f^*T_X \rightarrow N \rightarrow 0$, we get a map

$$\bigwedge^{n-4} N \rightarrow N^\vee \otimes f^*\mathcal{O}_X(n+1-d) \otimes \omega_S,$$

and from the short exact sequence $0 \rightarrow T_S \rightarrow f^*T_{\mathbf{P}^n} \rightarrow N' \rightarrow 0$, we get a map

$$\bigwedge^{n-3} N' \otimes f^*\mathcal{O}_X(-d) \rightarrow (N')^\vee \otimes f^*\mathcal{O}_X(n+1-d) \otimes \omega_S.$$

With the choices of the maps we have made, the following diagram, whose bottom row is obtained from dualizing sequence (1) and tensoring with $f^*\mathcal{O}_X(n+1-2d) \otimes \omega_S$, is commutative with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \bigwedge^{n-3} N \otimes f^*\mathcal{O}_X(-d) & \longrightarrow & \bigwedge^{n-3} N' \otimes f^*\mathcal{O}_X(-d) & \longrightarrow & \bigwedge^{n-4} N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & f^*\mathcal{O}_X(n+1-2d) \otimes \omega_S & \longrightarrow & (N')^\vee \otimes f^*\mathcal{O}_X(n+1-d) \otimes \omega_S & \longrightarrow & N^\vee \otimes f^*\mathcal{O}_X(n+1-d) \otimes \omega_S \longrightarrow 0 \end{array}$$

Since the cokernel of the first vertical map restricted to C is a torsion sheaf, to show the assertion, it will suffice to show that the map

$$H^1(S, \bigwedge^{n-3} N \otimes f^* \mathcal{O}_X(-d)) \rightarrow H^1(C, \bigwedge^{n-3} N \otimes f^* \mathcal{O}_X(-d)|_C)$$

is surjective. Applying the long exact sequence of cohomology to the top sequence, the surjectivity assertion follows if we show that

- (1) $H^0(S, \bigwedge^{n-4} N) \rightarrow H^0(C, \bigwedge^{n-4} N|_C)$ is surjective.
- (2) $H^1(C, \bigwedge^{n-3} N' \otimes f^* \mathcal{O}_X(-d)|_C) = 0$.

Since R is a sweeping component, $N_{f(C)/X}$ is globally generated, so $f^* T_X|_C$ and $N|_C$ are globally generated as well. Thus we can conclude (1) from the assumption that $\pi_* N$ is globally generated and hence $H^0(S, N) \rightarrow H^0(C, N|_C)$ is surjective.

To show (2), note that there is a surjective map $f^* \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus n+1} \rightarrow f^* T_{\mathbf{P}^n}$, so we get a surjective map $f^* \mathcal{O}_{\mathbf{P}^n}(1)^{\oplus n+1} \rightarrow N'$. Taking the $(n-3)$ -rd exterior power, and then tensoring with $f^* \mathcal{O}_X(-d)$, we get a surjective map

$$f^* \mathcal{O}_{\mathbf{P}^n}(n-3-d)^{\oplus \binom{n+1}{n-3}} \rightarrow \bigwedge^{n-3} N' \otimes f^* \mathcal{O}_X(-d).$$

Restricting to C , since $n-3-d \geq 0$, we have $H^1(C, \bigwedge^{n-3} N' \otimes f^* \mathcal{O}_X(-d)|_C) = 0$. \square

We conclude this section with a result on the case $d = n - 2$. Let C be a smooth rational curve of degree e in \mathbf{P}^n whose normal bundle N_{C/\mathbf{P}^n} is globally generated. If we write

$$N_{C/\mathbf{P}^n} = \mathcal{O}_C(a_1) \oplus \cdots \oplus \mathcal{O}_C(a_{n-1}),$$

then $\sum_{1 \leq i \leq n-1} a_i = e(n+1) - 2$. We say N_{C/\mathbf{P}^n} is *balanced* if $a_i + a_j < 3e$ for every $i \neq j$, and we call an irreducible component R of $R_e(X)$ a *balanced component* if N_{C/\mathbf{P}^n} is balanced for a general curve C parametrized by R .

If C is balanced and $d = n - 2$, then $H^1(C, \bigwedge^{n-3} N_{C/\mathbf{P}^n} \otimes \mathcal{O}_{\mathbf{P}^n}(-d)|_C) = 0$. Thus if N' is as in the proof of Theorem 1.1, we have

$$H^1(C, \bigwedge^{n-3} N' \otimes f^* \mathcal{O}_X(-d)|_C) = 0.$$

The proof of Theorem 1.1 then shows that

Proposition 3.2. *Suppose that X is a smooth hypersurface of degree $d = n - 2$ in \mathbf{P}^n . Let R be a sweeping component of $R_e(X)$ and C a curve parametrized by a general point of R . If N_{C/\mathbf{P}^n} is balanced, then R is not uniruled.*

It might be true that a general hypersurface of degree $n - 2$ in \mathbf{P}^n has a balanced component when $n \geq 6$ and $e \geq 1$. If $n = 5$ and $d = 3$, then the normal bundle of the curves parametrized by general points of $R_e(X)$ are not balanced when $e \geq 6$ is an even integer (see Proposition 4.2).

4. CUBIC FOURFOLDS

In this section, we prove Theorem 1.2, but before giving the proof, we first briefly explain the idea of the proof given in [3].

Let $X \subset \mathbf{P}^5$ be a general hypersurface of degree 3. Then by [1, Proposition 2.4], $R_e(X)$ is irreducible, and if we denote by $\overline{\mathcal{M}}_{0,0}(X, e)$ the Kontsevich moduli space of stable maps of degree e from curves of genus zero to X and by $\widetilde{M}_{0,0}(X, e)$ the corresponding coarse moduli space, then $\widetilde{M}_{0,0}(X, e)$ is birational to $R_e(X)$. Let \widetilde{M} a desingularization of $\overline{\mathcal{M}}_{0,0}(X, e)$.

Theorem 4.1 ([3], Theorem 1.2). *Let $X \subset \mathbf{P}^5$ be a general cubic hypersurface. There is a canonical section $\omega_e \in H^0(\widetilde{M}, \Omega_{\widetilde{M}}^2)$ with the following property:*

(a) *If e is odd, $e \geq 5$, and if p is a general point of \widetilde{M} , then ω_e induces a non-degenerate pairing on $T_{\widetilde{M},p}$.*

(b) *If e is even, $e \geq 6$, and if p is a general point of \widetilde{M} , then the linear map $T_{\widetilde{M},p} \rightarrow T_{\widetilde{M},p}^\vee$ induced by ω_e has a 1-dimensional kernel.*

If Y is a non-singular projective variety, and if Y has a non-zero 2-form ω such that for a general point $p \in Y$, the kernel of the map

$$T_{Y,p} \rightarrow T_{Y,p}^\vee$$

induced by the restriction of ω to p has dimension at most m , then the fiber of the MRC fibration of Y at a general point is at most m -dimensional [3, Lemma 1.4]. So the above theorem implies Theorem 1.2.

The proof of Theorem 4.1 is based on a general construction of differential forms on any desingularization of $\overline{\mathcal{M}}_{0,0}(X, e)$. Pulling back forms to the universal curve over $\overline{\mathcal{M}}_{0,0}(X, e)$, and then integrating along the fibers, one gets a linear map

$$H^{i+1}(X, \Omega_X^{j+1}) \rightarrow H^i(\overline{\mathcal{M}}_{0,0}(X, e), \Omega_{\overline{\mathcal{M}}_{0,0}(X, e)}^j).$$

When $i = 0$, this map gives j -forms on the moduli stack. Invoking the existence of a trace map then gives j -forms on any desingularization of the coarse moduli space ([3], Proposition 3.6). For a cubic threefold, $H^1(X, \Omega_X^3)$ is 1-dimensional, so from the above construction, one gets a natural 2-form ω_e on \widetilde{M} .

The next step is to compute the dimension of the kernel of ω_e restricted to a general point $p \in \widetilde{M}$. Note that if $C \subset X$ is the rational curve parametrized by p , then $T_{\widetilde{M},p} = H^0(C, N_{C/X})$, so $\omega_{e,p}$ gives a map $\delta : \wedge^2 H^0(C, N_{C/X}) \rightarrow \mathbf{k}$.

To prove Theorem 4.1, for a general smooth rational curve C of degree e on X , N_{C/\mathbf{P}^n} and $N_{C/X}$ are computed, and then, the corresponding pairing is described.

The proof of Theorem 1.2 given in this section has some similarities with the proof outlined above, but our method is local, and that enables us to avoid the technicalities and most of the computations involved in the proof presented in [3]. For our purpose, it is enough to compute N_{C/\mathbf{P}^n} for a general rational curve C of degree e on X .

Proposition 4.2 ([3], 7.1). *Let X be a general cubic fourfold, and let C be a general smooth rational curve of degree $e \geq 5$ on X . Then*

$$N_{C/\mathbf{P}^5} = \begin{cases} \mathcal{O}_C(\frac{3e-1}{2})^{\oplus 4} & \text{if } e \text{ is odd,} \\ \mathcal{O}_C(\frac{3e}{2})^{\oplus 2} \oplus \mathcal{O}_C(\frac{3e-1}{2})^{\oplus 2} & \text{if } e \text{ is even.} \end{cases}$$

Proof of Theorem 1.2. When $e \geq 5$ is odd, $R_e(X)$ is balanced by Proposition 4.2, so the assertion follows from Proposition 3.2.

Let now $e \geq 6$ be an even integer, and assume on the contrary that general fibers of the MRC fibration of $R_e(X)$ are at least 2-dimensional. Let S and f be as in Proposition 2.2, and let C be a general fiber of π . Set $N = N_f$ and $Q = N_{f,\pi}$. Then the following properties are satisfied.

(i) The composition of the maps

$$H^0(S, Q) \rightarrow H^0(S, Q|_C) \rightarrow H^0(C, N|_C)$$

is surjective.

(ii) The composition of the maps

$$H^0(S, Q \otimes I_C) \rightarrow H^0(C, Q \otimes I_C|_C) \rightarrow H^0(C, N \otimes I_C|_C)$$

is non-zero.

We show these lead to a contradiction.

Let Q' be the normal sheaf of the map $S \rightarrow \mathbf{P}^5$ relative to π . We have $Q|_C = N_{C/X}$ and $Q'|_C = N_{C/\mathbf{P}^5}$. Since $N_{X/\mathbf{P}^5} = \mathcal{O}_X(3)$, there is a short exact sequence

$$(2) \quad 0 \rightarrow Q \rightarrow Q' \rightarrow f^*\mathcal{O}_X(3) \rightarrow 0.$$

Taking exterior powers, we obtain the following short exact sequence

$$(3) \quad 0 \rightarrow \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3) \rightarrow \bigwedge^2 Q' \otimes f^*\mathcal{O}_X(-3) \rightarrow Q \rightarrow 0.$$

Since this sequence splits locally, its restriction to C is also a short exact sequence

$$(4) \quad 0 \rightarrow \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C \rightarrow \bigwedge^2 Q' \otimes f^*\mathcal{O}_X(-3)|_C \rightarrow Q|_C \rightarrow 0.$$

Let V be the image of the restriction map

$$\alpha : H^1(S, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)) \longrightarrow H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C).$$

We will show that our assumptions imply that V is of codimension at least 2 and that the image of the boundary map $H^0(C, Q|_C) \rightarrow H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C)$ is a subset of V . This is not possible since by Proposition 4.2,

$$\begin{aligned} H^1(C, \bigwedge^2 Q' \otimes f^*\mathcal{O}_X(-3)|_C) &= H^1(C, \bigwedge^2 N_{C/\mathbf{P}^5} \otimes f^*\mathcal{O}_X(-3)|_C) \\ &= H^1(C, \mathcal{O}_C(-2) \oplus \mathcal{O}_C(-1)^{\oplus 2} \oplus \mathcal{O}_C) \\ &= \mathbf{k}. \end{aligned}$$

Lemma 4.3. *The kernel of the map $f^*T_X \rightarrow Q$ is a line bundle which contains $\bigwedge^2 T_S \otimes \pi^*\Omega_{\mathbf{P}^1}$ as a subsheaf.*

Proof. The kernel of $f^*T_X \rightarrow Q$ is equal to the kernel of the map induced by π on the tangent bundles $T_S \rightarrow \pi^*T_{\mathbf{P}^1}$ which we denote by F

$$0 \rightarrow F \rightarrow T_S \rightarrow \pi^*T_{\mathbf{P}^1}.$$

Since F is reflexive, it is locally free on S , and it is clearly of rank 1. Also the composition of the maps

$$\bigwedge^2 T_S \otimes \pi^*\Omega_{\mathbf{P}^1} \rightarrow \bigwedge^2 T_S \otimes \Omega_S = T_S \rightarrow \pi^*T_{\mathbf{P}^1}$$

is the zero-map. So $\bigwedge^2 T_S \otimes \pi^*\Omega_{\mathbf{P}^1}$ is a subsheaf of F . \square

Given a section $r \in H^0(C, Q \times I_C|_C)$, we can define a map

$$\beta_r : H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C) \longrightarrow H^1(C, \omega_S|_C) = \mathbf{k}$$

as follows. The lemma above implies there is a generically injective map

$$\Psi : \bigwedge^3 Q \otimes f^*\mathcal{O}_X(-3) \otimes I_C \rightarrow \omega_S \otimes \pi^*T_{\mathbf{P}^1} \otimes I_C.$$

Restricting to C , we get a map

$$\Psi|_C : (\bigwedge^3 Q \otimes f^*\mathcal{O}_X(-3) \otimes I_C)|_C \rightarrow \omega_S|_C.$$

Also, r gives a map

$$\Phi_r : \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C \xrightarrow{\wedge r} \bigwedge^3 Q \otimes f^*\mathcal{O}_X(-3) \otimes I_C|_C,$$

and we define β_r to be the map induced by the composition $\Psi|_C \circ \Phi_r$. Note that β_r is non-zero if $r \neq 0$.

Lemma 4.4. *For sections $r, r' \in H^0(C, Q \otimes I_C|_C)$, $\ker(\beta_r) = \ker(\beta_{r'})$ if and only if r and r' are scalar multiples of each other.*

Proof. By Serre duality, it is enough to show that the images of the maps

$$H^0(C, I_C^\vee|_C) = H^0(C, \omega_S^\vee|_C \otimes \Omega_C) \xrightarrow[\beta_{r'}^\vee]{\beta_r^\vee} H^0(C, (\bigwedge^2 Q^\vee \otimes f^*\mathcal{O}_X(3))|_C \otimes \Omega_C)$$

are the same if and only if r and r' are scalar multiples of each other. Since $Q|_C = N_{C/X}$, we have $\bigwedge^3 Q|_C = \bigwedge^3 N_{C/X} = f^*\mathcal{O}_X(3) \otimes \Omega_C$, so

$$(\bigwedge^2 Q^\vee \otimes f^*\mathcal{O}_X(3))|_C \otimes \Omega_C = Q|_C,$$

and the map

$$\beta_r^\vee : H^0(C, I_C^\vee|_C) \rightarrow H^0(C, Q|_C)$$

is simply given by r . Similarly, $\beta_{r'}^\vee$ is given by r' . The lemma follows. \square

Recall that by definition, we have a short exact sequence

$$0 \rightarrow \pi^*T_{\mathbf{P}^1}|_C \rightarrow Q|_C \rightarrow N|_C \rightarrow 0,$$

and $\pi^*T_{\mathbf{P}^1}|_C = I_C^{-1}|_C$. If we tensor this sequence with $I_C|_C$, we get the following short exact sequence

$$0 \rightarrow \mathcal{O}_C \rightarrow Q \otimes I_C|_C \rightarrow N \otimes I_C|_C \rightarrow 0.$$

Let i be a non-zero section in the image of $H^0(C, \mathcal{O}_C) \rightarrow H^0(C, Q \otimes I_C|_C)$. Then i induces a map

$$\beta_i : H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C) \longrightarrow H^1(C, \omega_S|_C) = \mathbf{k}$$

as above. Let

$$\gamma : H^0(C, Q|_C) \rightarrow H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C)$$

be the connecting map in sequence (4).

Lemma 4.5. *We have*

- (a) $V \subset \ker \beta_i$.
- (b) $\text{image}(\gamma) \subset \ker \beta_i$.

Proof. (a) From the short exact sequence $0 \rightarrow T_S \rightarrow f^*T_X \rightarrow N \rightarrow 0$, we get a map

$$\bigwedge^2 N \otimes f^*\mathcal{O}_X(-3) \rightarrow \omega_S,$$

and so, there is a commutative diagram

$$\begin{array}{ccccc} H^1(S, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)) & \longrightarrow & H^1(S, \bigwedge^2 N \otimes f^*\mathcal{O}_X(-3)) & \longrightarrow & H^1(S, \omega_S) = 0 \\ \downarrow \alpha & & & & \downarrow \\ H^1(C, \bigwedge^2 Q \otimes f^*\mathcal{O}_X(-3)|_C) & \xrightarrow{\beta_i} & & \longrightarrow & H^1(C, \omega_S|_C). \end{array}$$

(b) Applying the long exact sequence of cohomology to the exact sequence

$$0 \rightarrow \bigwedge^2 N \otimes f^*\mathcal{O}_X(3) \rightarrow \bigwedge^2 N' \otimes f^*\mathcal{O}_X(-3) \rightarrow N \rightarrow 0,$$

we get a map

$$H^0(C, N|_C) \rightarrow H^1(C, \bigwedge^2 N \otimes f^*\mathcal{O}_X(-3)|_C).$$

The map $\beta_i \circ \gamma$ factors through

$$H^0(C, Q|_C) \rightarrow H^0(C, N|_C) \rightarrow H^1(C, \bigwedge^2 N \otimes f^*\mathcal{O}_X(-3)|_C) \rightarrow H^1(C, \omega_S|_C),$$

and we have a commutative diagram

$$\begin{array}{ccccc} H^0(S, N) & \longrightarrow & H^1(S, \bigwedge^2 N \otimes f^*\mathcal{O}_X(-3)) & \longrightarrow & H^1(S, \omega_S) = 0 \\ \downarrow & & & & \downarrow \\ H^0(C, Q|_C) & \longrightarrow & H^0(C, N|_C) & \longrightarrow & H^1(C, \omega_S|_C). \end{array}$$

Thus we can conclude the assertion from the fact that the restriction map $H^0(S, N) \rightarrow H^0(C, N|_C)$ is surjective, and so the image of the composition of the above maps is contained in the image of the restriction map $H^1(S, \omega_S) \rightarrow H^1(C, \omega_S|_C)$ which is zero. \square

In the following lemma we prove similar results for the sections of $Q \otimes I_C|_C$ which are restrictions of global sections of $Q \otimes I_C$.

Lemma 4.6. *If $\tilde{r} \in H^0(S, Q \otimes I_C)$, and if $r = \tilde{r}|_C$, then we have*

- (a) $V \subset \ker(\beta_r)$.
- (b) $\text{image}(\gamma) \subset \ker(\beta_r)$.

Proof. (a) Since r is the restriction of \tilde{r} to C , we get a commutative diagram

$$\begin{array}{ccc} H^1(S, \wedge^2 Q \otimes f^* \mathcal{O}_X(-3)) & \xrightarrow{\wedge \tilde{r}} & H^1(S, \wedge^3 Q \otimes f^* \mathcal{O}_X(-3) \otimes I_C) \xrightarrow{\Psi} H^1(S, \omega_S \otimes \pi^* T_{\mathbf{P}^1} \otimes I_C) = 0 \\ \downarrow \alpha & & \downarrow \\ H^1(C, \wedge^2 Q \otimes f^* \mathcal{O}_X(-3)|_C) & \xrightarrow{\beta_r} & H^1(C, \omega_S|_C). \end{array}$$

(b) Consider the short exact sequence

$$0 \rightarrow I_C^{-1}|_C \rightarrow Q|_C \rightarrow N|_C \rightarrow 0.$$

Since by property (i), $H^0(S, Q) \rightarrow H^0(C, N|_C)$ is surjective, to prove the statement, it is enough to show that for any non-zero u in the image of $H^0(C, I_C^{-1}|_C) \rightarrow H^0(C, Q|_C)$, we have $\gamma(u) \in \ker \beta_r$.

Consider the diagram

$$\begin{array}{ccccc} H^0(C, Q|_C) & \xrightarrow{\gamma} & H^1(C, \wedge^2 Q \otimes f^* \mathcal{O}_X(-3)|_C) & & \\ \wedge i \downarrow \quad \downarrow \wedge r & & \wedge i \downarrow \quad \downarrow \wedge r & \searrow \beta_r & \\ H^0(C, \wedge^2 Q \otimes I_C|_C) & \xrightarrow{\lambda} & H^1(C, \wedge^3 Q \otimes f^* \mathcal{O}_X(-3) \otimes I_C|_C) & \xrightarrow{\beta_i} & H^1(C, \omega_S|_C) \\ & & & \searrow \psi & \end{array}$$

where λ is obtained from applying the long exact sequence of cohomology to the third wedge power of sequence (2), and ψ is induced by the map $\Psi|_C$. Then we have

$$\begin{aligned} \beta_r \circ \gamma(u) &= \psi \circ \lambda(u \wedge r) \\ &= \psi \circ \lambda(-r \wedge i) \quad (\text{up to a scalar factor}) \\ &= \beta_i \circ \gamma(-r) \\ &= 0, \end{aligned}$$

where the last equality comes from the fact that $\gamma(H^0(C, Q|_C)) \subset \ker \beta_i$ by part (b) of Lemma 4.5. \square

Let now $\tilde{r}_0 \in H^0(S, Q \otimes I_C)$ be so that its image in $H^0(C, N \otimes I_C|_C)$ is non-zero. Such \tilde{r}_0 exists by property (ii). Then $r_0 := \tilde{r}_0|_C$ defines a map β_{r_0} . According to Lemma 4.4, $\ker \beta_{r_0} \neq \ker \beta_i$, so the codimension of $\ker \beta_i \cap \ker \beta_{r_0}$ is at least 2. On the other hand, by the previous lemmas, $\text{image}(\gamma) \subset \ker \beta_i \cap \ker \beta_{r_0}$. This is a contradiction since $\dim H^1(C, \wedge^2 Q' \otimes f^* \mathcal{O}_X(-3)|_C) = 0$.

□

5. LOW DEGREE HYPERSURFACES

Let $X \subset \mathbf{P}^n$ be a general hypersurface of degree $d \leq n/2$. Then by the main theorem of [6], $R_e(X)$ is irreducible of dimension $e(n+1-d) + (n-4)$. If $d^2 \leq n$, then by [4] and [11], X is rationally simply connected. This means that for every $e \geq 2$, the evaluation morphism

$$ev : \overline{\mathcal{M}}_{0,2}(X, e) \rightarrow X \times X$$

is surjective and a general fiber of ev is irreducible and rationally connected. In particular, $R_e(X)$ is rationally connected for $e \geq 2$. If $e = 1$, then $R_1(X)$ is the Fano variety of lines on X which is rationally connected if and only if $\binom{d+1}{2} \leq n$ [8, V.4.7].

As it was mentioned in the introduction, we expect that when d^2 is large compared to n , $R_e(X)$ is not uniruled. Here we outline a possible approach based, in part, on the results of the previous sections.

If $R_e(X)$ is uniruled, then there are S and f with the two properties given in Proposition 2.1. We can assume that the pair (S, f) is minimal in the sense that a component of a fiber of π which is contracted by f cannot be blown down. Let N be the normal sheaf of f , and let C be a general fiber of π with ideal sheaf I_C in S .

Denote by H the pullback of a hyperplane in \mathbf{P}^n to S , and denote by K a canonical divisor on S . From the exact sequences $0 \rightarrow T_S \rightarrow f^*T_X \rightarrow N \rightarrow 0$ and $0 \rightarrow f^*T_X \rightarrow f^*T_{\mathbf{P}^n} \rightarrow f^*\mathcal{O}_{\mathbf{P}^n}(d) \rightarrow 0$ we get

$$\begin{aligned} \chi(N \otimes I_C) &= (n+1)\chi(f^*\mathcal{O}_{\mathbf{P}^n}(1) \otimes I_C) - \chi(f^*\mathcal{O}_{\mathbf{P}^n}(d) \otimes I_C) - \chi(I_C) - \chi(T_S \otimes I_C) \\ &= (n+1)\left(\frac{(H-C) \cdot (H-C-K)}{2} + 1\right) - \frac{(dH-C) \cdot (dH-C-K)}{2} - 1 \\ &\quad - \frac{-C \cdot (-C-K)}{2} - 1 - (2K^2 - 14) \\ &= \frac{(n+1-d^2)}{2}H^2 - \frac{(n+1-d)}{2}H \cdot K - 2K^2 - (n+1-d)e + 14. \end{aligned}$$

We claim that $2H + 2C + K$ is base-point free and hence has a non-negative self-intersection number. By the main theorem of [10], if $2H + 2C + K$ is not base point free, then there exists an effective divisor E such that either

$$(2H + 2C) \cdot E = 1, E^2 = 0 \quad \text{or} \quad (2H + 2C) \cdot E = 0, E^2 = -1.$$

The first case is clearly not possible. In the second case, $H \cdot E = 0$, and $C \cdot E = 0$. So E is a component of one of the fibers of π which is contracted by f and which is a (-1) -curve. This contradicts the assumption that (S, f) is minimal. Thus $(2H + 2C + K)^2 \geq 0$. Also, since $H^1(S, f^*\mathcal{O}_X(-1)) = 0$,

$$H \cdot (H + K) = 2\chi(f^*\mathcal{O}_X(-1)) - 2 \geq -2,$$

so we can write

$$\begin{aligned}
\chi(N \otimes I_C) &= \frac{2n+2-d^2-d}{2}H^2 - (n-d-15)(e-1) - 2 \\
&\quad - 2(2H+2C+K)^2 - \frac{n-d-15}{2}(H \cdot (H+K) + 2) \\
&\leq \frac{2n+2-d^2-d}{2}H^2 - (n-d-15)(e-1) - 2,
\end{aligned}$$

and therefore $\chi(N \otimes I_C)$ is negative when $d^2 + d \geq 2n + 2$ and $n \geq 30$.

The Leray spectral sequence gives a short exact sequence

$$0 \rightarrow H^1(\mathbf{P}^1, \pi_*(N \otimes I_C)) \rightarrow H^1(S, N \otimes I_C) \rightarrow H^0(\mathbf{P}^1, R^1\pi_*(N \otimes I_C)) \rightarrow 0,$$

and by our assumption on S and f , $H^1(\mathbf{P}^1, \pi_*(N \otimes I_C)) = 0$. If we could choose S such that $H^0(\mathbf{P}^1, R^1\pi_*(N \otimes I_C)) = 0$, then we could conclude that $\chi(N \otimes I_C) \geq 0$ and hence $R_e(X)$ could not be uniruled for $d^2 + d \geq 2n + 2$ and $n \geq 30$.

We cannot show that for a general X , a minimal pair (S, f) as in Proposition 2.1 can be chosen so that $H^0(\mathbf{P}^1, R^1\pi_*(N \otimes I_C)) = 0$. However, we prove that when X is general, and (S, f) is minimal, for every $t \geq 1$,

$$H^0(\mathbf{P}^1, R^1\pi_*(N \otimes I_C \otimes f^*\mathcal{O}_X(t))) = 0.$$

We also show that if $t \geq 0$ and $f(C)$ is t -normal,

$$H^1(\mathbf{P}^1, \pi_*(N \otimes I_C \otimes f^*\mathcal{O}_X(t))) = 0.$$

These imply that $\chi(N \otimes I_C \otimes f^*\mathcal{O}_X(t))$ is non-negative when X is general and $f(C)$ is t -normal. To finish the proof of Theorem 1.4, we compute $\chi(N \otimes I_C \otimes f^*\mathcal{O}_X(t))$ directly and show that it is negative when the inequality in the statement of the theorem holds.

Proof of Theorem 1.4. Let X be a general hypersurface of degree d in \mathbf{P}^n . If $R_e(X)$ is uniruled, then there are S and f as in Proposition 2.1. Assume the pair (S, f) is minimal. Let N be the normal sheaf of f , and let C be a general fiber of π . Then $H^0(S, N) \rightarrow H^0(C, N|_C)$ is surjective. The restriction map $H^0(S, f^*\mathcal{O}_X(m)) \rightarrow H^0(C, f^*\mathcal{O}_X(m)|_C)$ is also surjective since $f(C)$ is m -normal, so the restriction map $H^0(S, N \otimes f^*\mathcal{O}_X(m)) \rightarrow H^0(C, N \otimes f^*\mathcal{O}_X(m)|_C)$ is surjective as well. Therefore,

$$H^1(\mathbf{P}^1, \pi_*(N \otimes f^*\mathcal{O}_X(m) \otimes I_C)) = 0.$$

Now let C be an arbitrary fiber of π , and let C^0 be an irreducible component of C . Then by Proposition 5.3, $f^*(T_X(t))|_{C^0}$ is globally generated for every $t \geq 1$, and hence $N \otimes f^*\mathcal{O}_X(t)|_{C^0}$ is globally generated too. So Proposition 5.1 shows that for every $t \geq 1$

$$H^0(\mathbf{P}^1, R^1\pi_*(N \otimes f^*\mathcal{O}_X(t) \otimes I_C)) = 0.$$

By the Leray spectral sequence,

$$\begin{aligned}
H^1(S, N \otimes f^*\mathcal{O}_X(m) \otimes I_C) &= H^1(\mathbf{P}^1, \pi_*(N \otimes f^*\mathcal{O}_X(m) \otimes I_C)) \\
&\quad \oplus H^0(\mathbf{P}^1, R^1\pi_*(N \otimes f^*\mathcal{O}_X(m) \otimes I_C)) \\
&= 0,
\end{aligned}$$

and therefore,

$$\chi(N \otimes f^* \mathcal{O}_X(m) \otimes I_C) \geq 0.$$

We next compute $\chi(N \otimes f^* \mathcal{O}_X(m) \otimes I_C)$. For an integer $t \geq 0$, set

$$a_t = \chi(N \otimes I_C \otimes f^* \mathcal{O}_X(t)).$$

We have

$$a_t = \chi(N \otimes I_C) + \frac{2t(n+1-d) + t^2(n-3)}{2} H^2 - \frac{t(n-5)}{2} H \cdot K - t(n-3)e.$$

So

$$a_t = \frac{b_t}{2} H^2 + \frac{c_t}{2} H \cdot K - 2K^2 + d_t,$$

where

$$\begin{aligned} b_t &= (n+1-d^2) + 2t(n+1-d) + t^2(n-3), \\ c_t &= -(n+1-d) - t(n-5), \end{aligned}$$

and

$$d_t = -t(n-3)e - (n+1-d)e + 14.$$

A computation similar to the computation in the beginning of this section shows that

$$\begin{aligned} a_t &= \frac{b_t - c_t}{2} H^2 - 2(2H + 2C + K)^2 + \frac{c_t + 16}{2} (H \cdot (H + K) + 2) + (d_t - c_t - 32 + 16e) \\ &\leq \frac{b_t - c_t}{2} H^2 + (d_t - c_t - 32 + 16e). \end{aligned}$$

Since

$$d_t - c_t - 32 + 16e = -(e-1)(n-15-d+t(n-3)) - 2t - 2,$$

and since $n-15-d+t(n-3) \geq 2n-d-18 \geq 0$ for $t \geq 1$ and $n \geq 12$, we get

$$a_t < \frac{b_t - c_t}{2} H^2.$$

When $d^2 + (2t+1)d \geq (t+1)(t+2)n+2$, $b_t < c_t$, and so $a_t < 0$. If we let $t = m$, we get the desired result. \square

Proposition 5.1. *If E is a locally free sheaf on S such that for every irreducible component C^0 of a fiber of π , $E|_{C^0}$ is globally generated, then $R^1 \pi_* E = 0$.*

Proof. By cohomology and base change [7, Theorem III.12.11], it suffices to prove that for every fiber C of π , $H^1(C, E|_C) = 0$. By Lemma 5.2, we can write $C = C_1 + \cdots + C_l$ such that every C_i is an irreducible component of C and such that $(C_1 + \cdots + C_i) \cdot C_{i+1} \leq 1$ for every $1 \leq i \leq l-1$. Hence

$$H^1(C_{i+1}, (E \otimes \mathcal{I}_{C_1 + \cdots + C_i})|_{C_{i+1}}) = 0 \text{ for every } 0 \leq i \leq l-1.$$

On the other hand, for every $0 \leq i \leq l-2$, we have a short exact sequence of \mathcal{O}_S -modules

$$0 \rightarrow E \otimes \mathcal{I}_{C_1 + \cdots + C_{i+1}}|_{C_{i+2} + \cdots + C_l} \rightarrow E \otimes \mathcal{I}_{C_1 + \cdots + C_i}|_{C_{i+1} + \cdots + C_l} \rightarrow E \otimes \mathcal{I}_{C_1 + \cdots + C_i}|_{C_{i+1}} \rightarrow 0.$$

So a decreasing induction on i shows that for every $0 \leq i \leq l - 2$,

$$H^1(S, E \otimes \mathcal{I}_{C_1 + \dots + C_i} |_{C_{i+1} + \dots + C_l}) = 0.$$

□

Lemma 5.2. *Let C be a fiber of π with l irreducible components counted with multiplicity. Then as a 1-cycle, C can be written as $C_1 + \dots + C_l$ such that each C_i is an irreducible component of C and for every $1 \leq i \leq l - 1$,*

$$(C_1 + \dots + C_i) \cdot C_{i+1} \leq 1.$$

Proof. The proof is by induction on l . If $l = 1$, there is nothing to prove. Assume the assertion holds for $k \leq l - 1$. There is at least one component C^0 of C such that $C^0 \cdot C^0 = -1$. Let r be the multiplicity of C^0 in C . Blowing down C^0 , we get a rational surface S' over \mathbf{P}^1 . Denote by C' the blow-down of C . Then by the induction hypothesis, we can write

$$C' = C'_1 + \dots + C'_{l-r}$$

such that $(C'_1 + \dots + C'_i) \cdot C'_{i+1} \leq 1$ for every $1 \leq i \leq l - r - 1$. Let C_i be the proper transform of C'_i . Then if in the above sum we replace C'_i by C_i when C_i does not intersect C^0 and by $C_i + C^0$ when C_i intersects C^0 , we get the desired result for C . □

Proposition 5.3. *Let $X \subset \mathbf{P}^n$ be a general hypersurface of degree d . For any morphism $h : \mathbf{P}^1 \rightarrow X$, $h^*(T_X(1))$ is globally generated.*

Proof. The proposition follows from [13, Proposition 1.1]. We give a proof here for the sake of completeness.

Consider the short exact sequence

$$0 \rightarrow h^*T_X \rightarrow h^*T_{\mathbf{P}^n} \rightarrow h^*\mathcal{O}_X(d) \rightarrow 0.$$

Since X is general, the image of the pull-back map $H^0(X, \mathcal{O}_X(d)) \rightarrow H^0(\mathbf{P}^1, h^*\mathcal{O}_X(d))$ is contained in the image of the map $H^0(\mathbf{P}^1, h^*T_{\mathbf{P}^n}) \rightarrow H^0(\mathbf{P}^1, h^*\mathcal{O}_X(d))$. Choose a homogeneous coordinate system for \mathbf{P}^n . Let p be a point in \mathbf{P}^1 , and without loss of generality assume that $h(p) = (1 : 0 : \dots : 0)$. We show that for any $r \in h^*(T_X(1))|_p$, there is $\tilde{r} \in H^0(\mathbf{P}^1, h^*(T_X(1)))$ such that $\tilde{r}|_p = r$.

Consider the exact sequence

$$0 \longrightarrow H^0(\mathbf{P}^1, h^*T_X(1)) \longrightarrow H^0(\mathbf{P}^1, h^*T_{\mathbf{P}^n}(1)) \xrightarrow{\phi} H^0(\mathbf{P}^1, h^*\mathcal{O}_X(d+1)).$$

Denote by s the image of r in $h^*(T_{\mathbf{P}^n}(1))|_p$. There exists $S \in H^0(\mathbf{P}^n, T_{\mathbf{P}^n}(1))$ such that the restriction of $\tilde{s} := h^*(S)$ to p is s . Denote by T the image of S in $H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(d+1))$, and let $\tilde{t} = h^*(T)$. Then T is a form of degree $d+1$ on \mathbf{P}^n , and since $\tilde{t}|_p = 0$, we can write

$$T = x_1G_1 + \dots + x_nG_n,$$

where the G_i are forms of degree d . Our assumption implies that for every $1 \leq i \leq n$, there is $\tilde{s}_i \in H^0(\mathbf{P}^1, h^*T_{\mathbf{P}^n})$ such that $\phi(\tilde{s}_i) = h^*G_i$. Then

$$\phi(\tilde{s} - h^*(x_1)\tilde{s}_1 - \dots - h^*(x_n)\tilde{s}_n) = \tilde{t} - h^*(x_1G_1) - \dots - h^*(x_nG_n) = 0,$$

and therefore, $\tilde{s} - h^*(x_1)\tilde{s}_1 - \dots - h^*(x_n)\tilde{s}_n$ is the image of some $\tilde{r} \in H^0(\mathbf{P}^1, h^*(T_X(1)))$. Since $(\tilde{s} - h^*(x_1)\tilde{s}_1 - \dots - h^*(x_n)\tilde{s}_n)|_p = \tilde{s}|_p = s$, we have $\tilde{r}|_p = r$.

□

We remark that although for every e and n with $e \geq n + 1 \geq 4$, there are smooth non-degenerate rational curves of degree e in \mathbf{P}^n which are not $(e - 3)$ -normal [5, Theorem 3.1], a general smooth rational curve of degree e in a general hypersurface of degree d has possibly a smaller normality.

For example, let C be a smooth rational curve of degree e in \mathbf{P}^n , and write

$$N_{C/\mathbf{P}^n} = \mathcal{O}_C(a_1) \oplus \cdots \oplus \mathcal{O}_C(a_{n-1}).$$

Then it follows from [5, Proposition 1.2] that if $e \geq 3$, and if

$$m = \max \{a_i + a_j, 1 \leq i < j \leq n - 1\} + 1 - 2e,$$

the curve C is m -normal. We expect that N_{C/\mathbf{P}^n} is as balanced as possible, i.e. $|a_i - a_j| \leq 1$ for every $1 \leq i, j \leq n - 1$, if C is a general smooth rational curve in a general hypersurface of degree $\leq n/2$ and if the degree of C is large enough compared to n .

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