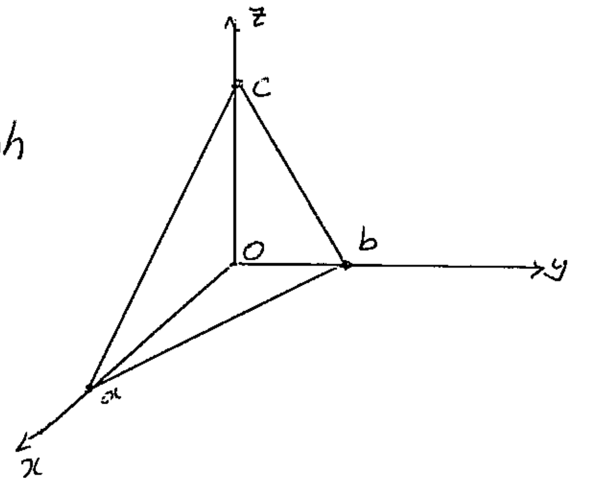
webwork: Problem Set 10, question # 11

Find the triple integral $\iiint_E xy\,dV$ where E is the solid tetrahedron with vertices $(a,0,0)$, $(0,b,0)$, $(0,0,c)$.



solution:

the equation of the plane passing through $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

the equation of the line on the xy-plane passing through $(a,0,0)$, $(0,b,0)$ is $\frac{x}{a}+\frac{y}{b}=1$,
so the limits of the region are given by:

$0 \le x \le a \qquad 0 \le y \le (1-\frac{x}{a})b \qquad 0 \le z \le c(1-\frac{x}{a}-\frac{y}{b})$

so

$$\iiint_E xy = \int_0^a \int_0^{b(1-\frac{x}{a})} \int_0^{c(1-\frac{x}{a}-\frac{y}{b})} xy\,dz\,dy\,dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} xyz \Big|_{z=0}^{c(1-\frac{x}{a}-\frac{y}{b})} dy\,dx = \int_0^a \int_0^{b(1-\frac{x}{a})} cx(1-\frac{x}{a})y - \frac{cx}{b}y^2\,dyd$$

$$= \int_0^a cx(1-\frac{x}{a})\frac{y^2}{2} - \frac{cx}{b}\frac{y^3}{3}\Big|_{y=0}^{b(1-\frac{x}{a})} dx = \int_0^a \frac{cb^2}{2}x(1-\frac{x}{a})^3 - \frac{cb^2}{3}x(1-\frac{x}{a})^3 d$$

$$= \frac{cb^2}{6}\int_0^a x(1-\frac{x}{a})^3 dx = \frac{cb^2}{6a^3}\int_0^a x(a-x)^3 dx = \frac{cb^2}{6a^3}\int_0^a (xa^3 - 3x^2a^2 + 3x^3a - x^4)\,dx$$

$$= \frac{cb^2}{6a^3}\left(a^3\frac{x^2}{2} - 3a^2\frac{x^3}{3} + 3a\frac{x^4}{4} - \frac{x^5}{5}\Big|_{x=0}^{x=a}\right) = \frac{cb^2}{6a^3}\left(\frac{a^5}{2} - a^5 + 3\frac{a^5}{4} - \frac{a^5}{5}\right)$$

$$= \frac{cb^2a^5}{6a^3}\cdot\left(\frac{30-60+45-12}{60}\right) = \frac{cb^2a^2}{6}\times\frac{1}{20} = \boxed{\frac{1}{120}cb^2a^2}$$